A SCHOOL GEOMETRY

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A

SCHOOL GEOMETRY

PARTS I AND II

Part I Lines and Angles Rectilineal Figures
Part II Areas of Rectilineal Figures

(Containing the substance of Euclid, Book I)

BY

H S. HALL, MA

AND

F H STEVENS MA

THIRD IDITION REVISED

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PREFACE

The present work provides a course of Elementary Geometry based on the recommendations of the Mathematical Association and on the schedule recently proposed and adopted at Cambridge

The principles which governed these proposals have been confirmed by the issue of revised schedules for all the more important Examinations, and they are now so generally accepted by teachers that they need no discussion here. It is enough to note the following points

- (1) We agree that a pupil should gain his first geometrical ideas from a short prelimmary course of a practical and experimental A suitable introduction to the present book would consist of Easy Exercises in Drawing to illustrate the subject matter of the Definitions, Measurements of Lines and Angles, Use of Compasses and Protractor, Problems on Bisection, Perpendiculars, and Parallels. Use of Set Squares. The Construction of Triangles and Quadrilaterals These problems should be accompanied by informal explanation, and the results verified by measurement Concurrently, there should be a senies of exercises in Driwing and Measurement designed to lead inductively to the more important Theorems of Part I [Enc. L 1-34].* While strongly advocating some such introductory lessons, we may point out that our book, as far as it goes, is complete in itself, and from the first is illustrated by numerical and graphical examples of the easiest types. Thus, throughout the whole work, a graphical and experimental course is provided side by side with the usual deductive exercises
- (11) Theorems and Problems are arranged in separate but parallel courses, intended to be studied para passu. This arrangement is made possible by the use, new generally sanctioned, of Hypothetical Constructions. These, before being employed in the text, are carefully specified, and referred to the Axiems on which they depend.

^{*}Such an introductory course is now furnished by our Lessons in Experimental and Practical Geometry

yı Preface

- (111) The subject is placed on the basis of Commensurable Magnitudes. By this means, certain difficulties which are wholly beyond the grasp of a young learner are postponed, and a wide field of graphical and numerical illustration is opened. Moreover the fundamental Theorems on Areas (hardly less than those on Proportion) may thus be reduced in number, greatly simplified, and brought into line with practical applications.
- (iv) An attempt his been made to curtail the excessive body of text which the demands of Examinations have hitherto forced as "bookwork" on a beginner's memory. Even of the Theorems here given a certain number (which we have distinguished with an asterisk) might be omitted or postponed at the discretion of the teacher. And the formal propositions for which—as such—teacher and pupil are held responsible, might perhaps be still further limited to those which make the landmarks of Elementary Geometry. Time so gained should be used in getting the pupil to apply his knowledge, and the working of examples should be made as important a part of a lesson in Geometry as it is so considered in Arithmetic and Algebra

Though we have not always followed Euclid's order of Propositions, we think it desirable for the present, in regard to the subject matter of Euclid Book I to preserve the essentials of his logical sequence. Our departure from Euclid's treatment of Areas has already been mentioned, the only other important divergence in this section of the work is the position of I 26 (Theorem 17), which we place after I 32 (Theorem 16), thus getting rid of the tedious and uninstructive Second Case. In subsequent Parts a freer treatment in respect of logical order has been followed.

As regards the presentment of the propositions, we have constantly kept in mind the needs of that large class of students, who, without special aptitude for mathematical study, and under no necessity for acquiring technical knowledge, may and do derive real intellectual advantage from lessons in pure deductive reasoning Nothing has as yet been devised as effective for this purpose as the Euclidean form of proof, and in our opinion no excuse is needed for treating the earlier proofs with that fulness which we have always found necessary in our experience as teachers

PPELACF VII

The examples are numerous and for the most part easy. They have been very carefully arranged, and are distributed throughout the text in immediate connection with the propositions on which they depend. A special feature is the large number of examples involving graphical or numerical work. The answers to these have been printed on perforated pages, so that they may easily be removed if it is found that access to numerical results is a source of temptation in examples involving measurement.

We are indebted to several friends for advice and singuistions. In particular we wish to express our thanks to Mr. H. C. Playing and Mr. H. C. Beaven of Chifton College for the valuable assistance they have rendered in reading the proof sheets and checking the answers to some of the numerical exercises.

H S HALL F H STEVENS

November, 1903

PREFATORY NOTE TO THE THIRD EDITION

In the present edition some further steps have been taken towards the curtailment of bookwork by reducing certain less important propositions (e.g. Euclid I. 22, 43, 44) to the rank of exercises Room has thus been found for more numerical and graphical excresses, and experimental work such as that leading to the Theorem of Pythagoras

Theorem 22 (page 62), in the shape recommended in the Cambridge Schedule, replaces the equivalent proposition given as Additional Theorem 1 (page 60) in previous editions

In the case of a few problems (e.g. Problems 23, 28, 29) it has been thought more instructive to justify the construction by a pre-liminary analysis than by the usual formal proof.

H S HALL F H STEVENS

March, 1904.

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GEOMETRY

PART L

AXIOMS

ALL mathematical reasoning is founded on certain simple principles, the truth of which is so evident that they are accepted without proof. These self-evident truths are called Axioms.

For instance

Things which are equal to the same thing are equal to one another

The following axioms, corresponding to the first four Rules of Arithmetic, are among those most commonly used in geometrical reasoning

Addition. If equals are added to equals, the sums are equal

Subtraction. If equals are taken from equals, the remainders are equal

Multiplication. Things which are the same multiples of equals are equal to one another

For instance Doubles of equals are equal to one another

Division. Things which are the same parts of equals are equal to one another

For instance Halres of equals are equal to one another

The above Axioms are given as instances, and not as a complete list, of those which will be used. They are said to be general, because they apply equally to magnitudes of all kinds. Certain special axioms relating to geometrical magnitudes only will be stated from time to time as they are required.

DEFINITIONS AND FIRST PRINCIPLES

Every beginner knows in a general way what is meant by a point, a line, and a surface But in geometry these terms are used in a strict sense which needs some explanation

1 A point has position, but is said to have no magnitude

This means that we are to attach to a point no idea of size either as to length or breadth, but to think only where it is situated. A dot made with a sharp pencil may be taken as roughly representing a point, but small as such a dot may be, it still has some length and breadth, and is therefore not actually a geometrical point. The smaller the dot however, the more nearly it represents a point.

2 A line has length, but is said to have no breadth

A line is traced out by a moving point. If the point of a peneil is moved over a sheet of paper, this trace left represents a line. But such a trace, however finely drawn, has some degree of breadth, and is therefore not itself a true geometrical line. The finer the trace left by the moving pencil point, the more nearly will it represent a line.

3 Proceeding in a similar manner from the idea of a line to the idea of a surface, we say that

A surface has length and breadth, but no thickness And finally,

A solid has length, breadth, and thickness

Solids, surfaces, lines and points are thus related to one another

- (1) A solid is bounded by surfaces
- (11) A surface is bounded by lines, and surfaces meet in lines
- (111) A line is bounded (or terminated) by points, and lines meet in points.

4 A line may be straight or curved

A straight line has the same direction from point to point throughout its whole length

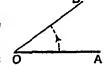
A curved line changes its direction continually from point to point

AXIOM There can be only one straight line joining two given points that is,

Two straight lines cannot enclose a space

- 5 A plane is a flat surface, the test of flatness being that if any two points are taken in the surface, the straight line between them lies wholly in that surface
- 6 When two straight lines meet at a point, they are said to form an angle

The straight lines are called the arms of the angle, the point at which they meet is its vertex.



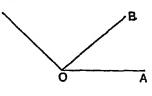
The magnitude of the angle may be thus of explained

Suppose that the arm OA is fixed, and that OB turns about the point O (as shewn by the arrow) Suppose also that OB began its turning from the position OA. Then the size of the angle AOB is measured by the amount of turning required to bring the revolving arm from its first position OA into its subsequent position OB

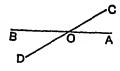
Observe that the size of an angle does not in any way depend on the length of its arms

Angles which he on either side of C a common arm are said to be adjacent

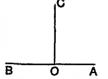
For example, the angles AOB, BOC, which have the common arm OB, are adjacent.



When two straight lines such as AB, CD cross one another at O, the angles COA, BOD are said to be vertically opposite. The angles AOD, COB are also vertically opposite to one another



7 When one straight line stands on another so as to make the adjacent angles equal to one another, each of the angles is called a right angle, and each line is said to be perpendicular to the other



AXIOMS (1) If O is a point in a straight line AB, then a line OC, which turns about O from the position OA to the position OB, must pass through one position, and only one, in which it is perpendicular to AB

(11) All right angles are equal

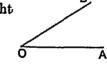
A right angle is divided into 90 equal parts called degrees (°), each degree into 60 equal parts called minutes ('), each minute into 60 equal parts called seconds (")

In the above figure, if OC revolves about O from the position OA into the position OB, it turns through two right angles, or 180°

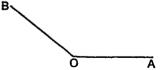
If OC makes a complete revolution about O, starting from OA and returning to its original position, it turns through four right angles, or 360°

8 An angle which is less than one right angle is said to be acute

That is, an acute angle is less than 90°



9 An angle which is greater than one right angle, but less than two right angles, is said to be obtuse



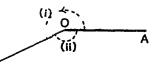
That is, an obtuse angle hes between 90° and 180°

10 If one arm OB of an angle turns until it makes a straight line with the other arm OA, the angle so formed is E called a straight angle



A straight angle=2 right angles=180°

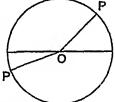
11 An angle which is greater than two right angles, but less than four right angles, is said to be reflex.



That is, a reflex angle hes between B 180° and 360°

Note. When two straight lines meet, too angles are formed, one greater, and one less than two right angles. The first arises by supposing OB to have revolved from the position OA the longer way round, marked (1), the other by supposing OB to have revolved the shorter way round, marked (11). Unless the contrary is stated, the angle between two straight lines will be considered to be that which is less than two right angles.

- 12 Any portion of a plane surface bounded by one or more lines is called a plane figure
- by a line traced out by a point which moves so that its distance from a certain fixed point is always the same



Here the point P moves so that its distance P from the fixed point O is always the same

The fixed point is called the centre, and the bounding line is called the circumference

- 14 A radius of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal
- 15 A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference
- ✓ 16 An arc of a circle is any part of the circumference

517 A semi-circle is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter



To bisect means to divide into two equal parts 18

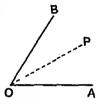
AXIOMS (1) If a point O moves from A to B along the straight line \overline{A} Ŕ AB, it must pass through one position in which it divides AB into two equal parts

That is to sav

Every finite straight line has a point of bisection

(11) If a line OP, revolving about O turns from OA to OB, it must pass through one position in which it divides the angle AOB into two equal parts

That is to say



Every angle may be supposed to have a line of bisection.

. HYPOTHETICAL CONSTRUCTIONS

From the Axioms attached to Definitions 7 and 18, it follows that we may suppose

- (1) A straight line to be drawn perpendicular to a given straight line from any point in it
 - (11) A finite straight line to be bisected at a point
 - (111) An angle to be bisected by a line

SUPERPOSITION AND EQUALITY

AXIOM Magnitudes which can be made to coincide with one another are equal

This axiom implies that any line, angle, or figure, may be taken up from its position, and without change in size or form, laid down upon a second line, angle, or figure, for the purpose of comparison, and it states that two such magnitudes are equal when one can be exactly placed over the other without overlapping

This process is called superposition, and the first magnitude is said to be applied to the other

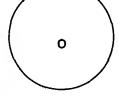
POSTULATES

In order to draw geometrical figures certain instruments are required. These are, for the purposes of this book, (1) a straight ruler (11) a pair of compasses. The following Postulates (or requests) claim the use of these instruments, and assume that with their help the processes mentioned below may be duly performed.

Let it be granted

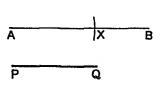
- In That a straight line may be drawn from any one point to any other point
- 2 That a FINITE (or terminated) straight line may be PRODUOED (that is, prolonged) to any length in that straight line
- 3 That a circle may be drawn with any point as centre and with a radius of any length

Notes (1) Postulate 3, as stated above, im plies that we may adjust the compasses to the length of any straight line PQ, and with a radius of this length draw a circle with any point O as centre. That is to say, the compasses may be used to transfer distances from one part of a diagram to another



(11) Hence from AB, the greater of two straight lines, we may cut off a part equal to PQ the less

For if with centre A, and radius equal to PQ, we draw an arc of a circle cutting AB at X, it is obvious that AX is equal to PQ.



INTRODUCTORY

- 1 Plane geometry deals with the properties of such lines and figures as may be drawn on a plane surface
- 2 The subject is divided into a number of separate discussions, called propositions

Propositions are of two kinds, Theorems and Problems

- A Theorem proposes to prove the truth of some geometrical statement
- A Problem proposes to perform some geometrical construction, such as to draw some particular line, or to construct some required figure
 - 3 A Proposition consists of the following parts

The General Enunciation, the Particular Enunciation, the Construction, and the Proof

- (1) The General Enunciation is a preliminary statement, describing in general terms the purpose of the proposition
- (11) The Particular Enunciation repeats in special terms the statement already made, and refers it to a diagram, which enables the reader to follow the reasoning more easily
- Y (111) The Construction then directs the drawing of such straight lines and circles as may be required to effect the purpose of a problem, or to prove the truth of a theorem
- in a problem has been accomplished, or that the property stated in a theorem is true
- 4 The letters QED are appended to a theorem, and stand for Quod erat Demonstrandum, which was to be proved

- 5 A Corollary is a statement the truth of which follows readily from an established proposition, it is therefore appended to the proposition as an inference or deduction, which usually requires no further proof
- 6 The following symbols and abbreviations are used in the text of this book

In Part I

for therefore, \triangle for angle, = ,, is, or are, equal to, \triangle ,, triangle

After Part I

pt for point, perp for perpendicular, st line , straight line, par , parallelogiam, rectil , rectilineal, par (or ||) , parallel, par , square, properties of perpendicular, par , parallelogiam, rectil , rectilineal, or circle, or , eircumference,

and all obvious contractions of commonly occurring words, such as opp, adj, diag, etc, for opposite, adjacent, diagonal, etc

[For convenience of oral work, and to prevent the rather common abuse of contractions by beginners, the above code of signs has been introduced gradually, and at first somewhat sparingly]

In numerical examples the following abbieviations will be used

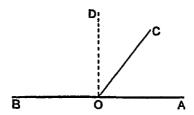
m for metre, cm for contimetie mm , millimetre km , kilometre

Also inches are denoted by the symbol (")
Thus 5" means 5 inches

ON LINES AND ANGLES

THEOREM 1 Euclid I 13]

The adjacent angles which one straight line makes with another straight line on one side of it, are together equal to two right angles



Let the straight line CO make with the straight line AB the adjacent \angle AOC, COB

It is required to prove that the LA AOC, COB are together equal to two right angles

Suppose OD is at right angles to BA.

Proof Then the L* AOC, COB together

= the three L* AOC, COD, DOB

Also the Lª AOD, DOB together

= the three L* AOC, COD, DOB

the \angle * AOC, COB together = the \angle * AOD, DOB = two right angles

QED

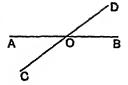
PROOF BY ROTATION

Suppose a straight line revolving about O turns from the position OA into the position OC, and thence into the position OB, that is, let the revolving line turn in succession through the \angle ^s AOC, COB

Now in passing from its first position OA to its final position OB, the revolving line turns through two right angles, for AOB is a straight line

Hence the Lª AOC, COB together = two right angles

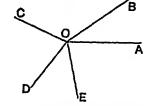
COROLLARY 1 If two straight lines cut one another, the four angles so formed are together equal to four right angles



For example,

∠BOD+∠DOA+∠AOC+∠COB=4 right angles

COROLLARY 2 When any number of straight lines meet at a point, the sum of the consecutive angles so formed is equal to four right angles



For a straight line revolving about O, and turning in succession through the L*AOB, BOC, COD, DOE, EOA, will have made one complete revolution, and therefore turned through four right angles

DEFINITIONS

(1) Two angles whose sum is two right angles, are said to be supplementary and each is called the supplement of the other

Thus in the Fig of Theor 1 the angles AOC, COB are supplementary Again the angle 123° is the supplement of the angle 57°

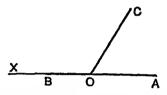
(11) Two angles whose sum 15 one right angle are said to be complementary, and each 18 called the complement of the other

Thus in the Fig of Theor 1 the angle DOC is the complement of the angle AOC Again angles of 34° and 56° are complementary

- COROLLARY 3 (1) Supplements of the same angle are equal
 - (n) Complements of the same angle are equal

THEOREM 2 [Euclid I 14]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line



At O in the straight line CO let the two straight lines OA, OB, on opposite sides of CO, make the adjacent L^* AOC, COB together equal to two right angles (that is, let the adjacent L^* AOC, COB be supplementary)

It is required to prove that OB and OA are in the same straight line

Produce AO beyond O to any point X it will be shewn that OX and OB are the same line

Proof Since by construction AOX is a straight line, the \(\ampli \) COX is the supplement of the \(\ampli \) COA Theor 1.

But, by hypothesis,

the LCOB is the supplement of the LCOA

the $\angle COX =$ the $\angle COB$, OX and OB are the same line

But, by construction, OX is in the same straight line with OA,

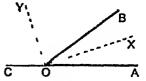
hence OB is also in the same straight line with OA

QED

EXERCISES

- 1 Write down the supplements of one half of a right angle, four thirds of a right angle, also of 16°, 149°, 83°, 101° 15'
- 2 Write down the complement of two fifths of a right angle, also of 27', 38 16, and 11° 29' 30"
- 3 If two straight lines intersect forming four angles of which one is known to be a right angle, prove that the other three are also right angles
- 4 In the triangle ABC the angles ABC, ACB are given equal If the side BC is produced both ways, show that the exterior angles so formed are equal
- 5 In the triangle ABC the angles ABC, ACB are given equal If AB and AC are produced beyond the base, show that the exterior angles so formed are equal
- The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the internal and external bisectors of the given angle

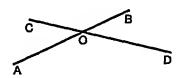
Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB.



- 6 Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle. That is to say, the internal aid external bisectors of an angle are at right angles to one arother.
- 7 Shew that the angles AOX and GOY in the above diagram are complementary
- 8 Shew that the angles BOX and COX are supplementary, and also that the angles AOY and BOY are supplementary
 - 9 If the angle AOB is 35°, find the angle COY

THEOREM 3 [Euclid I 15]

If two straight lines cut one another, the vertically opposite angles are equal



Let the straight lines AB, CD cut one another at the point O

It is required to prove that

- (1) the $\angle AOC = the \angle DOB$,
- (11) the $\angle COB = the \angle AOD$

Proof Because AO meets the straight line Cb, the adjacent \angle AOC, AOD together = two right angles, that is, the \angle AOC is the supplement of the \angle AOD

Again, because DO meets the straight line AB, the adjacent \angle DOB, AOD together = two right angles, that is, the \angle DOB is the supplement of the \angle AOD

Thus each of the \angle AOC, DOB is the supplement of the \angle AOD, the \angle AOC=the \angle DOB

Similarly, the $\angle COB = the \angle AOD$

QED

PROOF BY ROTATION

Suppose the line COD to revolve about O until OC turns into the position OA Then at the same moment OD must reach the position OB (for AOB and COD are straight)

Thus the same amount of turning is required to close the \angle AOC as to close the \angle DOB

the LAOC=the LDOB

EXERCISES ON ANGLES

(Numerical)

- 1 Through what angles does the minute hand of a clock turn in (1) 5 minutes, (11) 21 minutes, (111) 43½ minutes, (11) 14 min 10 sec ? And how long will it take to turn through (v) 66°, (vi) 222°?
- 2 A clock is started at noon through what angles will the hour hand have turned by (1) 3 45, (11) 10 minutes past 5. And what will be the time when it has turned through 172½°?
- 3 The earth makes a complete revolution about its axis in 24 hours Through what angle will it turn in 3 hrs 20 min , and how long will it take to turn through 130° ?
 - 4 In the diagram of Theorem 3
- (1) If the \triangle AOC=35°, write down (without measurement) the value of each of the \triangle *COB, BOD, DOA
- (11) If the ∠ªCOB, AOD together make np 250°, find each of the ∠ªCOA, BOD
- (111) If the L^a AOC, COB, BOD together make up 274°, find each of the four angles at O

(Theoretical)

- 5 If from O a point in AB two straight lines OC, OD are drawn on opposite sides of AB so as to make the angle COB equal to the angle AOD, shew that OC and OD are in the same straight line.
- √ 6 Two straight lines AB, CD eross at O If OX is the bisector of the angle BOD, prove that XO produced bisects the angle AOC
- 7 Two straight lines AB, CD cross at O If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.
- 8 If OX bisects an angle AOB, shew that, by folding the diagram about the bisector, OA may be made to coincide with OB

How would OA fall with regard to OB, if

- (1) the $\angle AOX$ were greater than the $\angle XOB$,
- (11) the LAOX were less than the LXOB?
- 9 AB and CD are straight lines intersecting at right angles at O , shew by folding the figure about AB, that OC may be made to fall along OD $\,$
- 10 A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB, shew that the crease left in the paper is perpendicular to AB $$^{\circ}$$

ON TRIANGLES

1 Any portion of a plane surface bounded by one or more lines is called a plane figure

The sum of the bounding lines is called the perimeter of the figure The amount of surface enclosed by the perimeter is called the area.

- 2 Rectilineal figures are those which are bounded by straight lines
- 3 A triangle is a plane figure bounded by three straight lines
- 4 A quadrilateral is a plane figure bounded by four straight lines
- 5 A polygon is a plane figure bounded by more than four straight lines



- 6 A rectilineal figure is said to be equilateral, when all its sides are equal, equiangular, when all its angles are equal, regular, when it is both equilateral and equiangular
- 7 Triangles are thus classified with regard to their sides A triangle is said to be

equilateral, when all its sides are equal, isosceles, when two of its sides are equal, scalene, when its sides are all unequal



Equilateral Triangle.



Isosceles Triangle.



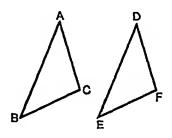
Scalene Triangle.

In a triangle ABC, the letters A, B, C often denote the magnitude of the several angles (as measured in degrees), and the letters a, b, c the lengths of the opposite sides (as measured in inches, centimetres, or some other unit of length)



THEOREM 4 [Euclid I 4]

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles included by those sides equal, then the triangles are equal in all respects



Let ABC, DEF be two triangles in which AB = DE.

AC = DF

and the included angle BAC = the included angle EDF

It is required to prove that the $\triangle ABC = the \stackrel{.}{\triangle} DEF$ in all respects

Proof

Apply the \triangle ABC to the \triangle DEF, so that the point A falls on the point P, and the side AB along the side DE

Then because AB = DE, the point B must coincide with the point E

And because AB falls along DE,
and the \(\text{BAC} = \text{the} \text{LEDF},
\)
AC must fall along DF

And because AC = DF, the point C must coincide with the point F

Then since B coincides with E, and C with F, the side BC must coincide with the side EF

Hence the \triangle ABC coincides with the \triangle DEF, and is therefore equal to it in all respects

Obs In this Theorem we must carefully observe what is given and what is proved

Given that
$$\begin{cases} AB = DE, \\ AC = DF, \\ and the \angle BAC = the \angle EDF \end{cases}$$

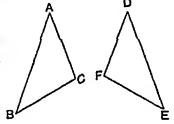
From these data we prove that the triangles coincide on superposition

Hence we conclude that
$$\begin{cases} BC = EF, \\ the \ \angle ABC = the \ \angle DEF, \\ and the \ \angle ACB = the \ \angle DFE, \end{cases}$$

also that the triangles are equal in area

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal

NOTE. The adjoining diagram shows that in order to make two congruent triangles coincide, it may be necessary to recerce, that is, turn over one of them before superposition



EXERCISES

- 1 Shew that the dissector of the vertical angle of an isosceles triangle (1) bisects the base (11) is perpendicular to the base.
- Let 0 be the middle point of a straight line AB, and let OC be perpendicular to 1' Then if P is any point in OC, prove that PA=PB
- 3 Assuming that the four sides of a square are equal, and that its angles are all right angles, prove that in the square ABCD, the diagonals AC, BD are equal
- 4 ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD prove that
 - (1) LM=MN
- (II) AM=DM
- (m) AN=AM

(iv) BN = DM

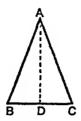
[Draw a separate figure in each case]

5 ABC is an isosceles triangle from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that BY=CX

20 GEOMETRY

THEOREM 5 [Euclid I 5]

The angles at the base of an isosceles triangle are equal



Let ABC be an isosceles triangle, in which the side AB = the side AC

It is required to prove that the $\angle ABC = the \angle ACB$

Suppose that AD is the line which bisects the \angle BAC, and let it meet BC in D

1st Proof Then in the \triangle BAD, CAD,

BA = CA

because $\begin{cases} AD \text{ is eommon to both triangles,} \\ and the included <math>\angle BAD = \text{the included } \angle CAD \end{cases}$,

the triangles are equal in all respects, Theo 4 so that the $\angle ABD = the \angle ACD$

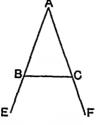
QFD

2nd Proof Suppose the \triangle ABC to be folded about AD Then since the \angle BAD = the \angle CAD, AB must fall along AC

And since AB = AC,
B must fall on C, and consequently DB on DC
the \angle ABD will coincide with the \angle ACD, and is therefore equal to it

Q E.D

COROLLARY 1 If the equal sides AB, AC of an isosceles triangle are produced, the exterior angles EBC, FCB are equal, for they are the supplements of the equal angles at the base



COROLLARY 2 If a triangle is equilateral, it is also equi-

DEFINITION A figure is said to be symmetrical about a line when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence

The straight line is called an axis of symmetry.

That this may be possible, it is clear that the two parts of the figuremust have the same size and shape, and must be similarly placed with regard to the axis

Theorem 5 proves that an isosceles triangle is symmetrical about the hiscolor of its VERTICAL angle

An equilateral triangle is symmetrical about the disector of ANY

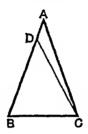
ONF of its angles

EXERCISES

- 1 ABCD is a four sided figure whose sides are all equal, and the diagonal BD is drawn show that
 - (1) the angle ABD=the angle ADB,
 - (11) the angle CBD=the angle CDB, (111) the angle ABC=the angle ADC
- 2. ABC, DBC are two isosceles triangles drawn on the same base BC, but on opposite sides of it—prove (by means of Theorem 5) that the angle ABD=the angle ACD
- BC and on the same side of it employ Theorem 5 to prove that the angle ABD=the angle ACD
- 4 AB, AC are the equal sides of an isosceles triangle ABC, and L, M, N are the middle points of AB, BC, and CA respectively prove that
 - (1) LM = NM (11) BN = CL
 - (111) the angle ALM = the angle ANM

THEOREM 6 [Euclid I 6]

If two angles of a triangle are equal to one another, then the sides which are opposite to the equal angles are equal to one another



Let ABC be a triangle in which the \angle ABC = the \angle ACB

It is required to prove that the side AC = the side AB

If AC and AB are not equal, suppose that AB is the greater From BA cut off BD equal to AC Join DC

Proof

Then in the \triangle DBC, ACB,

because {

BC is common to both, and the included \angle DBC = the included \angle ACB

the \triangle DBC = the \triangle ACB in area, Theor 4 the part equal to the whole, which is absurd

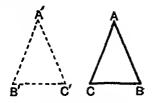
AB is not unequal to AC, that is, AB = AC

Q E.D

COROLLARY Hence if a triangle is equiangular it is also equilateral

NOTE ON THEOREMS 5 AND 6

Theorems 5 and 6 may be verified experimentally by entting out the given \triangle ABC, and, after turning it over, fitting it thus reversed into the vacant space left in the paper



Suppose A'B'C' to be the original position of the \triangle ABC, and let ACB represent the triangle when reversed

In Theorem 5, it will be found on applying A to A' that C may be

made to fall on B', and B on C

In Theorem 6, on applying C to B' and B to C' we find that A will

fall on A'

In either case the given triangle reversed will coincide with its own "trace," so that the side and angle on the left are respectively equal to the side and angle on the right

NOTE ON A THEORFY AND ITS CONVERSE



The enunciation of a theorem consists of two clauses. The first clause tells us what we are to assume, and is called the hypothesis, the second tells us what it is required to prove, and is called the conclusion.)

For example, the enunciation of Theorem 5 assumes that in a certain tringle ABC the side AB=the side AC this is the hypothesis From this it is required to prove that the angle ABC=the angle ACB this is the conclusion

to If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the converse of the first?

For example, in Theorem 5

it is assumed that

AB = AC

it is required to prove that the angle ABC = the angle ACB

Now in Theorem 6

it is assumed that the angle ABC = the angle ACB,

it is required to prove that AB=AC

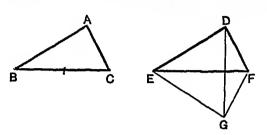
Thus we see that Theorem 6 is the converse of Theorem 5, (for the hypothesis of each is the conclusion of the other)

In Theorem 6 we employ an indirect method of proof frequently used in geometry (It consists in sliewing that the theorem cannot be untrue, since, if it were, we should be led to some impossible conclusion. This form of proof is known as Reductio ad Absurdum, and is most commonly used in demonstrating the converse of some foregoing theorem.)

It must not however be supposed that if a theorem is true, its converse is necessarily true. [See p. 25]

THEOREM 7 [Euclid I 8]

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they are equal in all respects



Let ABC, DEF be two triangles in which

AB = DE

AC = DF

BC = EF

It is required to prove that the triangles are equal in all respects

Proof
Apply the ABC to the ADEF,
so that B falls on E, and BC along EF, and
so that A is on the side of EF opposite to D.
Then because BC=EF, C must fall on F

Let GEF be the new position of the ABC Join DG

Because ED = EG, the $\angle EDG = the \angle EGD$

Theor 5

Again, because FD = FG, the $\angle FDG =$ the $\angle FGD$

Hence the whole \angle EDF = the whole \angle EGF, that is, the \angle EDF = the \angle BAC

Then in the \triangle BAC, EDF,

BA = ED, AC = DF,

and the included \angle BAC=the included \angle EDF,

the triangles are equal in all respects Theor 4
QED

Ob. In this Theorem

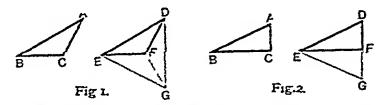
and we prove that $\triangle B = DE$, BC = EF, CA = FD, and we prove that $\triangle C = \triangle F$, $\triangle A = \triangle D$, $\triangle B = \triangle E$. Also the triangles are equal in area

Notice that the argies which are proved equal in the tiro triangles are opposite to sides which were given equal

Note 1 We have taken the case in which DG falls within the L. EDF, EGF

Two other cases might arise

- (1) DG might fall outside the L' EDF, EGF [as in Fig. 1].
- (11) DG might coincide with DF, FG fas in Fig 2].



These cases will arise only when the given triangles are obtuse angled or right-angled, and las will be seen re-cafter, not even then if we regir by choosing for superposition the greater side of the L ABC as in the diagram of page 24

Note 2. Two triangles are said to be equiangular to one another when the angles of one are respectively equal to the angles of the other

Here i two triargles lare the three e dee of one enterally equal to the three eides of the Ober, we triargles are equiangular to ens uno ver

The student should state the con error theorem, and snew by a dagram that the converse is not necessarily true.

** At the stage Problems 1-5 and 8 [see page 70] may conveniently be taken, the proofs affording good illustrations of the Identical Equality of Two Triangles

EXERCISES

On The Identical Equality of Two Triangles Theorems 4 and 7

(Theoretical)

- 1 Show that the straight line which joins the vertex of an isosceles triangle to the middle point of the base,
 - (1) bisects the vertical angle
- (11) is perpendicular to the base
- 2 If ABCD is a rhombus, that is, an equilateral foursided figure, shew, by drawing the diagonal AC, that
 - (1) the angle ABC=the angle ADC,
 - (11) AC bisects each of the angles BAD, BCD
- $\stackrel{\bigvee}{3}$ If in a quadrilateral ABCD the opposite sides are equal, namely AB=CD and AD=CB, prove that the angle ADC=the angle ABC
- 4 If ABC and DBC are two isosceles triangles drawn on the same base BC, prove (by means of Theorem 7) that the angle ABD=the angle ACD, taking (1) the ease where the triangles are on the same side of BC, (11) the ease where they are on opposite sides of BC
- j 5 If ABC, DBC are two isoseeles triangles drawn on opposite sides of the same base BC, and if AD be joined, prove that each of the angles BAC, BDC will be divided into two equal parts
- [-] 6 Shew that the straight lines which join the extremities of the base of an isosceles triangle to the middle points of the opposite sides, are equal to one another
- 7 Two given points in the base of an isosceles triangle are equidistant from the extremities of the base—show that they are also equidistant from the vertex
- $\sqrt{}$ 8 Shew that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral
- 9 ABC is an isosceles triangle having AB equal to AC, and the angles at B and C are bisected by BO and CO show that
 - (1) BO=CO,
- (n) AO bisects the angle BAC
- ₹'10 Shew that the diagonals of a rhombus [see Ex 2] bisect one another at right angles
- The equal sides BA, CA of an isosceles triangle BAC are produced beyond the vertex A to the points E and F, so that AE is equal to AF, and FB, EC are joined—shew that FB is equal to EC

EXPRCISES ON TRIANGLES

(Numerical and Graphical)

1 Draw a triangle ABC, having given a=2.0", b=2.1", c=1.3" Measure the angles, and find their sum

1

- In the triangle ABC, a=7.5 cm, b=7.0 cm, and c=6.5 cm Draw and measure the perpendicular from B on CA.
 - 3 Draw a triangle ABC, in which a=7 cm, b=6 cm, C=65°

How would you prove theoretically that any two triangles having these parts are alike in size and shape. Invent some experimental illustration

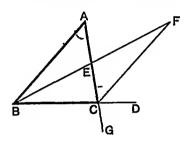
 $\sqrt{1}$ Drw a triangle from the following data $b=2^{\circ}$, $c=2.5^{\circ}$, A=57°, and measure a, B, and C

Draw a second triangle, using as data the values just found for a, B, and C and measure h, c, and A. What conclusion do you draw?

- 5 A Indder, whose foot is placed 12 feet from the base of a house, reaches to a window 35 feet above the ground. Draw a plan in which I represents 10 ft. and find by measurement the length of the ladder
- 6 I go due North 99 metres, then due East 20 metres. Plot my course (scale 1 cm. to 10 metres), and find by measurement as nearly as you can how for I am from my starting point.
- 7 When the sun is 12° above the horizon, a vertical pole casts a shadow 30 ft long. Represent this on a diagram (scale 1° to 10 ft) and find by measurement the approximate height of the pole
- S I rom a point A a surveyor goes 150 yards due Last to B then 300 yards due North to C finally 450 yards due North to D Plot his course (scale 1° to 100 yards), and find roughly how far D is from A Measure the augle DAB, and say in what direction D bears from A
- 9 B and C are two points I nown to be 260 yards apart, on a straight shore. A is a vessel at anchor. The angles CBA, BCA are observed to be 33 and \$1° respectively. Find graphically the approximate distance of the vessel from the points B and C, and from the nearest point on shore.
- 10 In surveying a parl it is required to hind the distance between two points A and B, but as a lake intervenes, a direct measurement cannot be made. The surveyor therefore takes a third point C, from which both A and B are necessible, and he finds CA=245 yards, CB=320 yards, and the angle ACB=42° Ascertain from a plan the approximate distance between A and B

THEOREM 8 Euchd I 16]

If one side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles



Let ABC be a triangle, and let BC be produced to D

It is required to prove that the exterior \angle ACD is greater than either of the interior opposite L. ABC, BAC

Suppose E to be the middle point of AC Join BE, and produce it to F, making EF equal to BE Join FC

Then in the A AEB, CEF, Proof

EB = EF, and the $\angle AEB = the$ vertically opposite $\angle CEF$, the triangles are equal in all respects. Theor 4 so that the $\angle BAE = the \angle ECF$

But the \angle ECD is greater than the \angle ECF, the LECD is greater than the LBAE, that is, the ACD is greater than the ABAC

In the same way, if AC is produced to G, by supposing A to be joined to the middle point of BC, it may be proved that the & BCG is greater than the & ABC

But the \angle BCG = the vertically opposite \angle ACD the ∠ACD is greater than the ∠ABC

COROLLARY 1 Any two angles of a triangle are together less han two right angles

For the LABC is less than the LACD Proved to each add the LACB

Then the L*ABC, ACB are less than the L*ACD, ACB, therefore less than two right angles

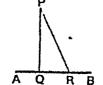


COROLLARY 2 Every triangle must have at least two acute nales

For if one angle is obtuse or a right angle, then by Cor 1 each of the ther nigles must be less than a right angle

Corollary 3 Only one perpendicular can be drawn to a tra-shiften from a oven point outsub-it

If two perpendiculars could be drawn to AB from 2, we should have a trivugle PQR in which each of he \$PQR, PRQ would be a right angle, which is mpossible.

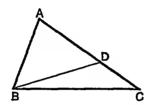


FITRCISES

- 1 Prove Corollers 1 by joining the vertex A to any point in the prove BC
- 2. ABC is a triangle aid D any point within it. If BD and CD are sound, the angle BDC is greater than the angle BAC. Prove this
 - (i) by producing BD to meet AC
 - (ii) by joining AD, and producing it towards the base
- If any side of a trimicle is produced both ways, the exterior ngles so formed are together greater than two right angles
- 4 To a given straight line there cannot be drawn from a point outide it more than to o straight lines of the same given length
- 5 If the equal sides of an isosceles triangle are produced, the exterior angles must be obtained

THEOREM 9 [Euclid I. 18]

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less



Let ABC be a triangle, in which the side AC is greater than the side AB

It is required to prove that the LABC is greater than the LACB

From AC cut off AD equal to AB
Join BD

Proof

Because AB = AD,

the $\angle ABD = the \angle ADB$

Theor 5

But the exterior \angle ADB of the \triangle BDC is greater than the interior opposite \angle DCB, that is, greater than the \angle ACB

• the \angle ABD is greater than the \angle ACB

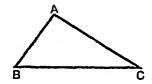
Still more then is the \angle ABC greater than the \angle ACB

QED

Obs The mode of demonstration used in the following Theorem is known as the <u>Proof by Exhaustion</u>. It is applicable to cases in which one of certain suppositions must necessarily be true, and it consists in shewing that each of these suppositions is filse with one exception hence the truth of the remaining supposition is inferred

THEOREM 10 [Euchd I 19]

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less



Let ABC be a triangle, in which the \angle ABC is greater than the \angle ACB

It is required to prove that the side AC is greater than the side AB

Proof

If AC is not greater than AB,
it must be either equal to, or less than AB

Now if AC were equal to AB, then the \angle ABC would be equal to the \angle ACB, Theor 5 but, by hypothesis, it is not

Again, if AC were less than AB, then the \angle ABC would be less than the \angle ACB, Theo 9 but, by hypothesis, it is not

That is, AC is neither equal to, nor less than AB

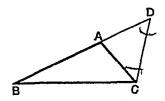
AC is greater than AB

QED

[For Exercises on Theorems 9 and 10 see page 34.]

THEOREM 11 [Euclid I 20]

Any two sides of a triangle are together greater than the third side



Let ABC be a triangle

It is required to prove that any two of its sides are together greater than the third side

It is enough to shew that if BC is the greatest side, then BA, AC are together greater than BC.

Produce BA to D, making AD equal to AC Join DC

Proof.

Because AD = AC, the $\angle ACD =$ the $\angle ADC$

Theor 5

But the \angle BCD is greater than the \angle ACD, the \angle BCD is greater than the \angle ADC, that is, than the \angle BDC

Hence from the \triangle BDC,

BD is greater than BC

Theo 10

But BD = BA and AC together, BA and AC are together greater than BC

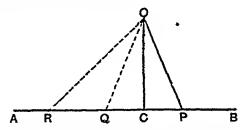
QED

Note This proof may serve as an evereise, but the truth of the Theorem is really self evident. For to go from B to C along the straight line BC is clearly shorter than to go from B to A and then from A to C. In other words

The shortest distance between two points is the straight line which joins a them.

THEOREM 12

Of all straight lines drawn from a given point to a given straight line the perpendicular is the least



Let OC be the perpendicular, and OP any oblique, drawn from the given point O to the given straight line AB

It is required to prove that OC is less than OP

Proof In the \triangle OCP, since the \angle OCP is a right angle, the \angle OPC is less than a right angle, Theor 8 Cor that is, the \angle OPC is less than the \angle OCP OC is less than OP Theor 10.

_.....

QED

COROLLARY 1 Hence conversely, since there can be only one perpendicular and one shortest line from O to AB,

If OC is the shortest straight line from O to AB, then OC is perpendicular to AB

COROLLARY 2 Two obliques OP, OQ, which cut AB at equal distances from C the foot of the perpendicular, are equal

The \triangle OCP, OCQ may be shown to be congruent by Theorem 4, hence OP=OQ.

COROLLARY 3 Of two obliques OQ, OR, if OR cuts AB at the greater distance from C the foot of the perpendicular, then OR is greater than OQ.

The LOQC is acute, the LOQR is obtuse, the LOQR is greater than the LORQ, OR is greater than OQ.

EXERCISES ON INEQUALITIES IN A TRIANGLE

- 1 The hypotenuse is the greatest side of a right angled triangle.
- 2 The greatest side of any triangle makes acute angles with each of the other sides
 - 3 If from the ends of a side of a triangle, two straight lines are drawn to a point within the triangle, then these straight lines are together less than the other two sides of the triangle
 - 4 BC, the base of an isoseeles triangle ABC, is produced to any point D, shew that AD is greater than either of the equal sides
 - 5 If in a quadrilatoral the greatest and least sides are opposite to one another, then each of the angles adjacent to the least side is greater than its opposite angle
 - 6 In a triangle ABC, if AC is not greater than AB, shew that any straight line drawn through the vertex A and terminated by the base BC, is less than AB
 - 7 ABC is a triangle, in which OB, OC bisect the angles ABC, ACB respectively show that, if AB is greater than AC, then OB is greater than OC
 - '8 The difference of any two sides of a triangle is less than the third side
- 9 The sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter
 - 10 The permeter of a quadrilateral is greater than the sum of its diagonals
 - 11 ,ABC is a triangle, and the vertical angle BAC is bisected by a line which meets BC in X, shew that BA is greater than BX, and CA greater than CX Honee obtain a proof of Theorem 11
 - 12 The sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle
- 13 The snm of the diagonals of a quadrilateral is less than the sum of the four straight lines drawn from the angular points to any given point. Prove this, and point out the exceptional case
- 14 In a triangle any two sides are together greater than twice the median which bisects the remaining side

[Produce the median, and complete the construction after the manner of Theorem 8]

15 In any triangle the sum of the medians is less than the perimeter

GRaj Bahadur MATHUR. \$3

PARALLELS

Parallel straight lines are such as, being in he same plane, do not meet however far they are produced peyond both ends

Note. Parallel lines must be in the same plane For instance, two straight lines, one of which is drawn on a table and the other on the loor would never meet if produced, but they are not for that reason necessarily parallel

AXIOM. Two intersecting straight lines cannot both be parallel 'o a third straight line

In other words

Through a given point there can be only one straight line parallel 'o a quen straight line

This assumption is known as Playfair's Axiom

When two straight lines AB, CD are met by DEFINITION 1 third straight line EF, eight angles are formed, to which for the sake of distinction particular names are given.

Thus in the adjoining figure, 1, 2, 7, 8 are called exterior angles, 3, 4, 5, 6 are called interior angles, 4 and 6 are said to be alternate angles, so also the angles 3 and 5 are alternate

to one another

Of the angles 2 and 6, 2 is referred to as the exterior angle, and 6 as the interior opposite angle on the same side

c

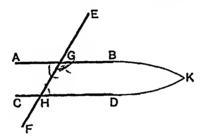
of EF Such angles are also known as corresponding angles Similarly 7 and 3, 8 and 4, 1 and 5 are pairs of corresponding angles

THEOREM 13 [Euclid I 27 and 28]

If a straight line cuts two other straight lines so as to make

- (1) the alternate angles equal,
- or (11) an exterior angle equal to the interior opposite angle on the same side of the cutting line,
- or (111) the interior angles on the same side equal to two right angles,

then in each case the two straight lines are parallel



(1) Let the straight line EGHF cut the two straight lines AB, CD at G and H so as to make the alternate L* AGH, GHD equal to one another

It is required to prove that AB and CD are parallel

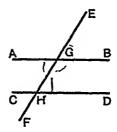
Proof If AB and CD are not parallel, they will meet, if produced, either towards B and D, or towards A and C

If possible, let AB and CD, when produced, meet towards B and D, at the point K

Then KGH is a triangle, of which one side KG is produced to A, the exterior \angle AGH is greater than the interior opposite \angle GHK, but, by hypothesis, it is not greater \mathcal{I}_{1} .

AB and CD cannot meet when produced towards B and D Similarly it may be shewn that they cannot meet towards A and C

AB and CD are parallel



(11) Let the exterior \angle EGB = the interior opposite \angle GHD It is required to prove that AB and CD are parallel

Proof
Because the \(\subseteq EGB = \text{the } \times QHD,
and the \(\subseteq EGB = \text{the vertically opposite } \times AGH,
the \(\times AGH = \text{the } \times GHD

and these are alternate angles,
AB and CD are parallel

(m) Let the two interior \angle * BGH, GHD be together equal to two right angles

It is required to prove that AB and CD are parallel

Proof Because the L*BGH, GHD together = two right angles, and because the adjacent L*BGH, AGH together = two right angles,

the L* BGH, AGH together = the L* BGH, GHD

From these equals take the \angle BGH, then the remaining \angle AGH = the remaining \angle GHD and these are alternate angles,

AB and CD are parallel

QED

DEFINITION A straight line drawn across a set of given lines is called a transversal

For instance, in the above diagram the line EGHF, which crosses the given lines AB, CD is a transversal

THEOREM 14 [Euclid I 29]

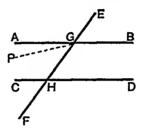
If a straight line cuts two parallel lines, it makes

(1) the alternate angles equal to one another,

(ii) the exterior angle equal to the interior opposite angle on the same side of the cutting line,

(111) the two interior angles on the same side together equal to two

right angles



Let the straight lines AB, CD be parallel, and let the straight line EGHF cut them.

It is required to prove that

- (1) the $\angle AGH = the alternate \angle GHD$,
- (11) the exterior $\angle EGB = the interior opposite \angle GHD$,
- (111) the two interior L. BGH, GHD together = two right angles

Proof (1) If the \angle AGH is not equal to the \angle GHD, suppose the \angle PGH equal to the \angle GHD, and alternate to it, then PG and CD are parallel Theor 13

But, by hypothesis, AB and CD are parallel, the two intersecting straight lines AG, PG are both parallel to CD which is impossible. Playfair's Axiom

the \angle AGH is not unequal to the \angle GHD, that is, the alternate \angle " AGH, GHD are equal

(11) Again, because the $\angle EGB = the$ vertically opposite $\angle AGH$,

and the \angle AGH = the alternate \angle GHD, Proved the exterior \angle EGB = the interior opposite \angle GHD

(111) Lastly, the \angle EGB = the \angle GHD, Proved add to each the \angle BGH, then the \angle EGB, BGH together = the angles BGH, GHD But the adjacent \angle EGB, BGH together = two right angles, the two interior \angle BGH, GHD together = two right angles of ED

PARALLELS ILLUSTRATED BY ROTATION

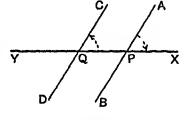
The direction of a straight line is determined by the angle which it makes with some given line of reference

Thus the direction of AB, relatively to the given line YX, is given by the angle APX

Now suppose that AB and CD in the adjoining diagram are parallel, then we have learned that the cxt $\angle APX$ =the int opp $\angle CQX$, that is, AB and CD make equal angles with the line of reference YX

This brings us to the leading idea connected with parallels Parallel straight lines have the same DIRFOTION, but differ in POSITION

The same idea may be illustrated thus



Suppose AB to rotate about P through the LAPX, so as to take the position XY Thence let it rotate about Q the opposite way through the equal LXQC it will now take the position CD Thus AB may be brought into the position of CD by two rotations which, being equal and opposite, involve no final change of direction

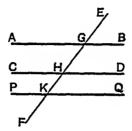
HYPOTHFTICAL CONSTRUCTION In the above diagram let AB be a fixed straight line, Q a fixed point, CD a straight line turning about Q, and YQPX any transversal through Q Then as CD rotates, there must be one position in which the \angle CQX = the fixed \angle APX

Hence through any given point we may assume a line to pass parallel to any given straight line

Obs If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite senses of the line AB

THEOREM 15 [Euclid L 30]

Straight lines which are parallel to the same straight line are parallel to one another



Let the straight lines AB, CD be each parallel to the straight line PQ.

It is required to prove that AB and CD are parallel to one another

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K

Proof Then because AB and PQ are parallel, and EF meets them,

the $\angle AGK =$ the alternate $\angle GKQ$.

And because CD and PQ are parallel, and EF meets them, the exterior \angle GHD = the interior opposite \angle GKQ \nearrow /

the \angle AGH = the \angle GHD, and these are alternate angles, AB and CD are parallel

QED

Note. If PQ has between AB and CD, the Proposition needs no proof, for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another

The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse

For if AB and CD were not parallel, they would meet when produced Then there would be two intersecting straight lines both parallel to a third straight line which is impossible

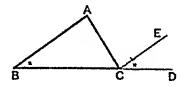
Therefore AB and CD never meet, that is, they are parallel

EXERCISES ON PARALLELS

- 1 In the diagram of the previous page, if the angle EGB is 55°, express in degrees each of the angles GHO, HKQ, QKF
- $\sqrt{-2}$. Straight lines which are perpendicular to the same straight line are -arphi parallel to one another
- If a straight line meet two or more parallel straight lines, and is sperpendicular to one of them, it is also perpendicular to all the others
- \supset 4. Angles of which the arms are parallel, each to each, are either \prec equal or supplementary
- 5 Two straight lines AB, CD bisect one another at O Shew that a the straight lines joining AC and BD are parallel
- 6 Any straight line drawn parallel to the base of an isosceles tri A angle makes equal angles with the sides
- 7 If from any point in the bisector of an angle a straight line is drawn parallel to either arm of the angle, the triangle thus formed is isosceles
- s From X, a point in the base BC of an isosceles triangle ABC, a straight line is drawn at right angles to the base, cutting AB in Y, and CA produced in Z shew the triangle AYZ is isosceles
- 9 If the straight line which bisects an exterior angle of a triangle is parallel to the opposite side, show that the triangle is isosceles
- The straight lines drawn from any point in the bisector of an angle parallel to the arms of the angle, and terminated by them, are equal and the resulting figure is a rhombus
- 11 AB and CD are two straight lines intersecting at D, and the adjacent angles so formed are bisected if through any point X in DC a straight line YXZ is drawn parallel to AB and meeting the bisectors in Y and Z, show that XY is equal to XZ
- 12 Two strught rods PA, QB revolve about pivots at P and Q, PA making 12 complete revolutions a minute, and QB making 10 If they start parallel and pointing the same way, how long will it be before they are again parallel, (i) pointing opposite ways, (ii) pointing the same way?

THFORFW 16 [Enchd I 32]

The three angles of a triangle are together equal to two right angles



Let ABC be a trinigle

It is required to prove that the three _'ABC, BCA, CAB together = two right angles

Produce BC to any point D, and suppose CE to be the line through C parallel to BA.

Proof Because BA and CE are parallel and AC meets them, the \angle ACE - the alternate \angle CAB

Again, because BA and CE are parallel, and BD meets them the exterior a ECD = the interior opposite a ABC

the whole exterior $\angle ACD = the sum of the two interior opposite <math>\angle^*CAB$, ABC

To each of these equals add the \(\mathcal{L} \) BCA then the \(\mathcal{L}^{\circ} \) BCA, ACD together = the three \(\mathcal{L}^{\circ} \) BCA, CAB, ABC

But the adjacent L' BCA, ACD together = two right angles the L' BCA, CAB, ABC together = two right angles OED

Obs In the course of this proof the following most important property has been established

If a side of a triangle is produced the exterior angle is equal to the sum of the two interior opposite angles

Namely, the ext LACD = the LCAB + the LABC

INFERENCES FROM THEOREM 16

I If A, B, and C denote the number of degrees in the angles of a triangle,

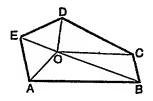
then $A+B+C=180^{\circ}$

- 12 If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other
- J 3 'In any right-angled triangle the two acute angles are complementary
- J4 If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled
- 5 The sum of the angles of any quadrilateral figure is equal to four right angles

EXERCISES ON THEOREM 16

- √1 Each angle of an equilateral triangle is two thirds of a right angle, or 60°
- $\sqrt{2}$. In a right-angled isosceles triangle each of the equal angles is 45°
- √3 Two angles of a triangle are 36° and 123° respectively deduce the third angle, and verify your result by measurement
- 4 In a triangle ABC, the $\angle B=111^{\circ}$, the $\angle C=42^{\circ}$, deduce the $\angle A$, and verify by measurement.
 - 5 One side BC of a triangle ABC is produced to D If the exterior angle ACD is 134°, and the angle BAC is 42°, find each of the remaining interior angles
 - 6 In the figure of Theorem 16, if the $\angle ACD = 118^{\circ}$, and the $\angle B = 51^{\circ}$, find the $\angle A$ and C, and check your results by measurement
- '7 Prove that the three angles of a triangle are together equal to two right angles by supposing a line drawn through the vertex parallel to the base
- 8 If two straight lines are perpendicular to two other straight lines, each to each, the acute angle between the first pair is equal to the acute angle between the second pair

COROLLARY 1 All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides



Let ABCDE be a rectilineal figure of n sides

It is required to prove that all the interior angles +4 rt L^{\bullet} = 2n rt L^{\bullet}

Take any point O within the figure, and join O to each of its vertices

Then the figure is divided into n triangles

And the three L^n of each Δ together = 2 rt L^n . Hence all the L^n of all the Δ^n together = 2n rt L^n .

But all the \angle ^{*} of all the \triangle ^{*} make up all the interior angles of the figure together with the angles at 0, which = 4 rt \angle ^{*}

all the int \angle ⁵ of the figure + 4 rt \angle ⁵ = 2n rt \angle ⁶

QED

DEFINITION A regular polygon is one which has all its sides equal and all its angles equal

Thus if D denotes the number of degrees in each angle of a regular polygon of n sides, the above result may be stated thus

$$nD + 360^{\circ} = n \quad 180^{\circ}$$

EXAMPLE

Find the number of degrees in each angle of

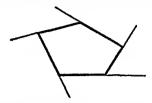
- (1) a regular hexagon (6 sides),
- (11) a regular octagon (8 sides).
- (111) a regular decagon (10 sides)

EXFRCISES ON THEOREM 16

(Numerical and Graphical)

- 1 ABC is a triangle in which the angles at B and C are respectively double and troble of the angle at A find the number of degrees in each of these angles
- 1/2 Express in degrees the angles of an isosecles triangle in which
 - (1) Each base angle is double of the vertical angle,
 - (ii) Each base angle is four times the vertical angle
- √3 The base of a triangle is produced both ways, and the exterior angles are found to be 94° and 126°, deduce the vertical angle Construct such a triangle, and check your result by measurement
- 4 The sum of the angles at the base of a triangle is 162', and their difference is 60° find all the angles
- √5 The angles at the base of a triangle are 84° and 62°, deduce (i) the vertical angle (ii) the angle between the bisectors of the base angles. Check your results by construction and measurement
- 56 In a triangle ABC, the angles at B and C are 74° and 62°, if AB and AC are produced, deduce the angle between the bisectors of the exterior angles. Check your result graphically
- 77 Three angles of a quadrilateral are respectively 1141°, 50°, and 751°, find the fourth angle
- Js In a quadrilateral ABCD the angles at B, C, and D are respectively equal to 2A, 3A, and 4A, find all the angles
- 70 Four angles of an irregular pentagon (5 sides) are 40°, 78°, 122°, and 135°, find the fifth angle
- 10 In any regular polygon of n sides, each angle contains $\frac{2(n-2)}{n}$ right angles
 - (1) Deduce this result from the Enunciation of Corollary I
- (ii) Prove it independently by joining one vertex A to each of the others (except the two immediately adjacent to A), thus dividing the polygon into n-2 triangles
- 11 How many sides have the regular polygons each of whose angles is (1) 105°, (11) 156°?
- 12 Shew that the only regular figures which may be fitted together so as to form a plane surface are (1) equilateral triangles, (11) squares, (111) regular hexagons

COROLLARY 2 If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then all the exterior ingles so formed are together equal to four right angles.



1st Proof. Suppose, as before, that the figure has n sides. and consequently n vertices

Now at each vertex

the interior \angle + the exterior \angle = 2 rt \angle . and there are n vertices,

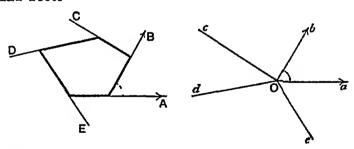
the sum of the int \angle^* + the sum of the ext $\angle^* = 2n$ rt \angle^*

But by Corollary 1,

the sum of the int L.+ 4 16 L" $=2n \text{ rt } \angle$, the sum of the ext $\angle^a = 4$ rt \angle^a

QED

2nd Proof



Take any point O, and suppose Oa, Ob, Oc, Od, and Oe, are lines parallel to the sides marked, A, B, C, D, E (and drawn from O in the sense in which those sides were produced)

Then the exterior \angle between the sides A and $B = \text{the } \angle aOb$

And the other exterior $\angle =$ the $\angle = bOc$, cOd, dOe, eOd, respectively

the sum of the ext \angle * = the sum of the \angle * at O =4 rt. /*

THE ANGLES OF RECTILINEAL FIGURES

EXERCISES >

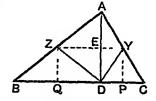
- '' 1 If one side of a regular hexagon is produced, shew that the exterior angle is equal to the interior angle of an equilateral triangle.
- 2 Express in degrees the magnitude of each exterior angle of (1) a regular octagon, (1) a regular decagon
- 3 How many sides has a regular polygon if each exterior angle is (1) 30°, (11) 24° 7
- 4 If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected, shew that the bisectors meet at right angles
- 5 If the base of any triangle is produced both ways, shew that the sum of the two exterior angles minus the vertical angle is equal to two right angles
- 6 In the triangle ABC the base angles at B and C are bisected by BO and CO respectively Shew that the angle BOC= $90^{\circ} + \frac{A}{2}$
- 7 In the triangle ABC, the sides AB, AC are produced, and the exterior angles are bisected by BO and CO Shew that the angle BOC= 90° - $\frac{A}{5}$
- 8 The angle contained by the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the remaining angles
- ' 9 A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to BA, if DC is drawn, shew that BCD is a right angle
- 10 The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

EXPERIMENTAL PROOF OF THEOREM 16 [A+B+C=180°]

In the \triangle ABC, AD is perp to BC the greatest side AD is bisected at right angles by ZY, and YP, ZQ are perp on BC

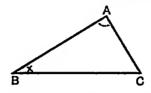
If now the \triangle is folded about the three dotted lines, the \angle * A, B, and C will coin ede with the \angle * ZDY, ZDQ, YDP.

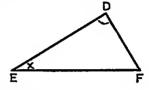
their sum is 180°



THEOREM 17 [Euclid I. 26]

If two triangles have two angles of one equal to two angles of the other, each to each, and any side of the first equal to the corresponding side of the other, the triangles are equal in all respects





Let ABC, DEF be two triangles in which the $\angle A =$ the $\angle D$, the $\angle B =$ the $\angle E$, also let the side BC = the corresponding side EF

It is required to prove that the \triangle ABC, DEF are equal in all respects

Proof The sum of the \angle * A, B, and C = 2 rt \angle * Theor 16 = the sum of the \angle * D, E, and F, and the \angle * A and B = the \angle * D and E respectively.

the $\angle C$ = the $\angle F$

Apply the \triangle ABC to the \triangle DEF, so that B falls on E, and BC along EF

Then because BC = EF, C must coincide with F

And because the $\angle B =$ the $\angle E$, BA must fall along ED

And because the $\angle C =$ the $\angle F$, CA must fall along FD

the point A, which falls both on ED and on FD, must coin cide with D, the point in which these lines intersect

the \triangle ABC coincides with the \triangle DEF, and is therefore equal to it in all respects

So that AB = DE, and AC = DF, and the \triangle ABC = the \triangle DEF in area. QED

EXERCISES

ON THE IDENTICAL EQUALITY OF TRIANGLES

- 1 Show that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal
- 2 Any point on the bisector of an angle is equidistant from the arms of the angle
- I 3 Through O, the middle point of a straight line AB, any straight line is drawn, and perpendiculars AX and BY are dropped upon it from A and B shew that AX is equal to BY
- 4 If the insector of the vertical angle of a triangle is at right angles to the base, the triangle is isoscoles
- 5 If m a triangle the perpendicular from the vertex on the base bisects the base, then the triangle is isosceles
- 6 If the bisector of the vertical angle of a triangle also bisects the base, the triangle is iso-celes

[Produce the bisector, and complete the construction after the manner of Theorem 8]

- 7 The middle point of any straight line which meets two parallel straight lines, and is terminated by them, is equidistant from the pirallels
- 8 A straight line drawn between two parallels and terminated by them, is bisected, show that any other straight line passing through the iniddle point and terminated by the parallels, is also bisected at that point
- 9 If through a point equidistant from two parallel straight lines, two straight lines are drawn cutting the parallels, the portions of the latter thus intercepted are equal
- 10 In a quadrulateral, ABCD, if AB=AD, and BC=DC show that the diagonal AC bisects each of the angles which it joins, and that AC is perpendicular to BD
- 11 A surveyor wishes to ascertain the breadth of a river which he cannot cross. Standing at a point A near the bank, he notes an object B immediately opposite on the other bank. He lays down a line AC of any length at right angles to AB, fixing a mark at O the middle point of AC from C he walks along a line perpendicular to AC until he reaches a point D from which O and B are seen in the same direction. He now measures CD prove that the result gives him the width of the river

ON THE IDENTICAL EQUALITY OF TRIANGLES

Three cases of the congruence of triangles have been dealt with in Theorems 4, 7, 17, the results of which may be summarised as follows

Two triangles are equal in all respects when the following three parts in each are severally equal

Two sides, and the included angle

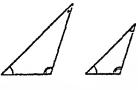
Theorem 4 Theorem 7

12 The three sides

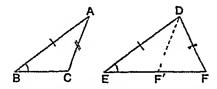
- Two angles and one side, the side given in one triangle CORRESPONDING to that onen in the other Theorem 17
- rwo triangles are not, however, necessarily equal in all respects when any three parts of one are equal to the corresponding parts of the other

For example

(1) When the three angles of one are equal to the three angles of the other, each to each, the adjoining diagrim shows that the triangles need not be equal in all respects



(n) When two sides and one angle in one are equal to two sides and one angle of the other, the given angles being opposite to equal sides, the diagram below shews that the triangles need not be equal in all respects



For if AB = DE, and AC = DF, and the $\angle ABC = the \angle DEF$, it will be seen that the shorter of the given sides in the triangle DEF may he in either of the positions DF or DF'

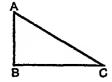
Note From these data it may be shewn that the angles opposite to the equal sides AB, DE are either equal (as for instance the LACB, DF'E) or supplementary (as the L. ACB, DFE), and that in the former case the triangles are equal in all respects. This is called the ambiguous case in the congruence of triangles [See Problem 9, p 82]

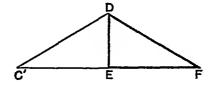
If the given angles at B and E are right angles, the ambiguity dis

This exception is proved in the following Theorem

THEOREM 18

Two right-angled triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects





Let ABC, DEF be two right-angled triangles, in which the L*ABC, DEF are right angles, the hypotenuse AC = the hypotenuse DF, and AB = DE

It is required to prove that the \triangle * ABC, DEF are equal in all respects

Proof Apply the ABC to the ADEF, so that AB falls on the equal line DE, and C on the side of DE opposite to F

Let C' be the point on which C falls.

Then DEC' represents the ABC in its new position

Since each of the \angle * DEF, DEC' is a right angle, EF and EC' are in one straight line

And in the $\triangle C'DF$, because DF = DC' (i.e. AC), the $\triangle DFC = \text{the } \triangle DC'F$ Theor 5

Hence in the \triangle DEF, DEC',

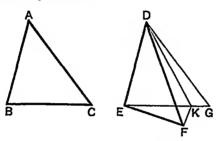
because $\begin{cases} \text{the } \angle \, \mathsf{DEF} = \mathsf{the} \, \angle \, \mathsf{DEC'}, \text{ being right angles }, \\ \text{the } \angle \, \mathsf{DFE} = \mathsf{the} \, \angle \, \mathsf{DC'E}, \\ \text{and the side DE is common} \end{cases}$

the \triangle DEF, DEC' are equal in all respects, Theor 17. that is, the \triangle DEF, ABC are equal in all respects

Q E.D

THEOREM 19 [Euclid I 24]

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle included by the two sides of one greater than the angle included by the corresponding sides of the other, then the base of that which has the greater angle is greater than the base of the other



Let ABC, DEF be two triangles, in which BA = ED, and AC = DF,

but the \angle BAC is greater than the \angle EDF

It is required to prove that the base BC is greater than the base EF

Proof Apply the \triangle ABC to the \triangle DEF, so that A falls on D, and AB along DE

Then because AB = DE, B must coincide with E Let DG, GE represent AC, CB in their new position Then if EG passes through F, EG is greater than EF, that is, BC is greater than EF

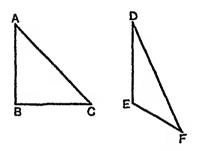
But if EG does not pass through F, suppose that DK bisects the \angle FDG, and meets EG in K Join FK

Then in the \triangle * FDK, GDK,

because $\begin{cases} FD = GD, \\ DK \text{ is common to both,} \\ and \text{ the included } \angle FDK = \text{the included } \angle GDK, \end{cases}$

FK = GK Theor 4.

Now the two sides EK, KF are greater than EF, that is, EK, KG are greater than EF EG (or BC) is greater than EF Q.E.D. Conversely, if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other, then the angle contained by the sides of that which has the greater base, is greater than the angle contained by the corresponding sides of the other



Let ABC, DEF be two triangles in which

BA = ED,

and AC = DF,

but the base BC is greater than the base EF

It is required to prove that the L BAC is greater than the LEDF.

Proof If the \angle BAC is not greater than the \angle EDF, it must be either equal to, or less than the \angle EDF

Now if the \(\text{BAC}\) were equal to the \(\text{EDF},\)
then the base BC would be equal to the base EF \(\text{Theor 4}\)
but, by hypothesis, it is not.

Again, if the LBAC were less than the LEDF, then the base BC would be less than the base EF, Theor 19 but, by hypothesis, it is not

That is, the \(\text{BAC} is neither equal to, nor less than the \(\text{EDF} \), the \(\text{BAC} is greater than the \(\text{EDF} \)

QED

^{*} Theorems marked with an asterisk may be omitted or postponed at the discretion of the teacher

REVISION LESSON ON TRIANGLES

- 1 State the properties of a triangle relating to
 - (1) the sum of its interior angles,
 - (11) the sum of its exterior angles

What property corresponds to (1) in a polygon of n sides? With what other figures does a triangle share the property (11)?

- 2 Classify triangles with regard to their angles Enunciate any Theorem or Corollary assumed in the classification
- 3 Enunciate two Theorems in which from data relating to the sides a conclusion is drawn relating to the augles

In the triangle ABC, if $\alpha=3.6$ cm, b=2.8 cm, c=3.6 cm, arrange the angles in order of their sizes (before measurement), and prove that the triangle is acute-angled.

4 Ennuciate two Theorems in which from data relating to the angles a conclusion is drawn relating to the sides

In the triangle ABC, if

- (1) A=48° and B=51°, find the third angle, and name the greatest side
- (11) A=B=62½°, find the third angle, and arrange the sides in order of their lengths
- 5 From which of the conditions given below may we conclude that the triangles ABC, A'B'C' are identically equal? Point out where ambiguity arises, and draw the triangle ABC in each case

(1)
$$\begin{cases} A = A' = 71^{\circ} \\ B = B = 46^{\circ} \\ \alpha = \alpha' = 3.7 \text{ cm} \end{cases}$$
 (1)
$$\begin{cases} \alpha = \alpha' = 4.2 \text{ cm} \\ b = b' = 2.4 \text{ cm} \\ C = C = 81^{\circ} \end{cases}$$
 (11)
$$\begin{cases} A = A' = 36^{\circ} \\ B = B' = 121^{\circ} \\ C = C' = 23^{\circ} \end{cases}$$

(1v)
$$\begin{cases} a = a = 3 \ 0 \text{ cm} \\ b = b' = 5 \ 2 \text{ cm} \\ c = c' = 4 \ 5 \text{ cm} \end{cases}$$
 (v)
$$\begin{cases} B = B' = 53^{\circ} \\ b = b' = 4 \ 3 \text{ cm}. \\ c = c' = 5 \ 0 \text{ cm} \end{cases}$$
 (vi)
$$\begin{cases} C = C = 90^{\circ} \\ c = c' = 5 \text{ cm} \\ a = a' = 3 \text{ cm} \end{cases}$$

- 6 Summarise the results of the last question by stating generally under what conditions two triangles
 - (1) are necessarily congruent,
 - (11) may or may not be congruent.
- 7 If two triangles have their angles equal, each to each, the triangles are not necessarily equal in all respects, because the three data are not independent. Carefully explain this statement.

(Miscellaneous Examples)

- 8 (1) The perpendicular is the shortest line that can be drawn to a given straight line from a given point
- (11) Obliques which make equal angles with the perpendicular are equal
- (111) Of two obliques the less 14 that which makes the smaller angle with the perpendicular
- 9 If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are equal in all respects
- 10 PQ is a perpendicular (4 cm in length) to a straight line XY Draw through P a series of obliques making with PQ the angles 15°, 30°, 45°, 60°, 75° Measure the lengths of these obliques, and tabulate the results
- 11 PAB is a triangle in which AB and AP have constant lengths 4 cm and 3 cm. If AB is fixed, and AP rotates about A, trace the changes in PB, as the angle A increases from 0° to 180°

Answer this question by drawing a series of figures, increasing A by increments of 30° Measure PB in each case, and tabulate the results

- 12 From B the foot of a flagstaff AB a horizontal line is drawn passing two points C and D which are 27 feet apart. The angles BCA and BDA are 65° and 40° respectively. Represent this on a diagram (scale 1 cm to 10 ft), and find by measurement the approximate height of the flagstaff.
- 13 From P, the top of a lighthouse PQ two boats A and B are seen at anchor in a line due south of the lighthouse. It is known that PQ=126 ft, \angle PAQ=57°, \angle PBQ=33°, hence draw a plan in which 1" represents 100 ft, and find by measurement the distance between A and B to the nearest foot
- 14 From a lighthouse L two slips A and B, which are 600 yards apart, are observed in directions S W and 15° East of South respectively. At the same time B is observed from A in a S E direction Draw a plan (scale 1" to 200 yds), and find by measurement the distance of the lighthouse from each slip

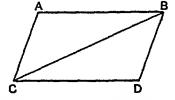
PARALLELOGRAMS

DEFINITIONS

1 A quadrilateral is a plane figure bounded by four straight lines The straight line which joins opposite angular
points in a quadrilateral is called a diagonal
/2 A parallelogram is a quadrilateral whose opposite sides are parallel
[It will be proved hereafter that the opposite sides of a parallelogram are equal, and that its opposite angles are equal]
' 3 A rectangle is a parallelogram which has one of its angles a right angle
[It will be proved hereafter that all the angles of a rectangle are right angles See page 59]
4 A square is a rectangle which has two advacent sides equal
[It will be proved that all the sides of a square are equal and all its angles right angles See page 59]
+ 5 A rhombus is a quadrilateral which has all its sides equal, but its angles are not right angles
6 A trapezium is a quadrilateral which has one pair of parallel sides

THEOREM 20 [Euchd I 33]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel



Let AB and CD be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AC and BD

It is required to prove that AC and BD are equal and parallel

Join BC

Proof Then because AB and CD are parallel, and BC meets them,

the $\angle ABC =$ the alternate $\angle DCB$

Now in the \triangle ABC, DCB,

because $\begin{cases} AB = DC, \\ BC \text{ is common to both }, \\ and \text{ the } \angle ABC = \text{the } \angle DCB, \end{cases}$

Proved

the triangles are equal in all respects,

so that AC = DB,

(1)

and the $\angle ACB = \angle DBC$

But these are alternate angles,
AC and BD are parallel

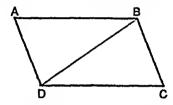
(11)

That is, AC and BD are both equal and parallel

QED

THEOREM 21 [Euclid I 31]

The opposite sules and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram



Let ABCD be a parallelogram, of which BD is a chagonal It is required to prove that

- (1) AB = CD, and AD = CB,
- (ii) the \(\mathbb{BAD} = \text{the} \(\text{DOB},
- (iii) the LADC = the LOBA,
- (iv) the \triangle ABD = the \triangle CDB in area

Proof Because AB and DC are parallel, and BD meets them, the \angle ABD = the alternate \angle CDB

Again, because AD and BC are parallel, and BD meets them, the \angle ADB = the alternate \angle CBD

Hence in the A"ABD, CDB,

because $\begin{cases} \text{the } \angle \, \mathsf{ABD} = \mathsf{the} \, \angle \, \mathsf{CDB}, \\ \mathsf{the} \, \angle \, \mathsf{ADB} = \mathsf{the} \, \angle \, \mathsf{CBD}, \\ \mathsf{and} \, \, \mathsf{BD} \, \mathsf{is} \, \mathsf{common} \, \mathsf{to} \, \mathsf{both} \end{cases}$

the triangles are equal in all respects, Theor 17 so that AB = CD, and AD = CB, (1) and the $\angle BAD =$ the $\angle DCB$, (11) and the $\triangle ABD =$ the $\triangle CDB$ in area (11)

And because the $\angle ADB =$ the $\angle CBD$, Proved and the $\angle CDB =$ the $\angle ABD$,

. the whole $\angle ADC =$ the whole $\angle CBA$ (m)

COROLLARY 1 If one angle of a parallelogram is a right angle, all its angles are right angles

In other words

All the angles of a rectangle are right angles

For the sum of two consecutive $\angle = 2$ rt. $\angle = 1$, (Theor 14) if one of these is a rt angle, the other must be a rt angle And the opposite angles of the parm are equal. all the angles are rt angles

All the sides of a square are equal, and all COROLLARY 2 us angles are right angles

COROLLARY 3 The diagonals of a parallelogram bisect one another

Let the diagonals AC, BD of the parm ABCD intersect at O

To prove AO=OC, and BO=OD

In the \triangle AOB, COD. (the LOAB=thealt LOCD. because the LAOB=vert. opp LCOD. and AB=the opp side CD,

OA=OC, and OB=OD Theor 17

EXERCISES

- If the opposite sides of a quadrilateral are equal, the figure is a parallelogram
- If the opposite angles of a quadrilateral are equal, the figure is a parallelogram
- If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram
 - 4 The diagonals of a rhombus bisect one another at right angles
- If the diagonals of a parallelogram are equal, all its angles are right angles
- In a parallelogram which is not rectangular the diagonals are unequal

EXERCISES ON PARALLELS AND PARALLELOGRAMS

(Symmetry and Superposition)

1 Show that by folding a rhombus about one of its diagonals the triangles on opposite sides of the crease may be made to coincide

That is to say, prove that a rhombus is symmetrical about either

diagonal

- 2 Prove that the diagonals of a square are axes of symmetry Name two other lines about which a square is symmetrical
- 3 The diagonals of a rectangle divide the figure into two congruent triangles is the diagonal, therefore, an axis of symmetry? About what two lines is a rectangle symmetrical?
- 4. Is there any axis about which an oblique parallelogram is symmetrical? Give reasons for your answer
- 5 In a quadrilateral ABCD, AB=AD and CB=CD, but the sides are not all equal Which of the diagonals (if oither) is an axis of symmetry?
 - 6 Prove by the mothod of superposition that
- (1) Two parallelograms are identically equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each, and one angle of one equal to one angle of the other
- (11) Two rectangles are equal if two adjacent sides of one are equal to two adjacent sides of the other, each to each
- 7 Two quadrilaterals ABCD, EFGH have the sides AB, BC, CD, DA equal respectively to the sides EF, FG, GH, HE, and have also the angle BAD equal to the angle FEH Shew that the figures may be made to coincide with one another

(Miscellaneous Theoretical Examples)

- 8 Any straight line drawn through the middle point of a diagonal of a parallelogram and terminated by a pair of opposite sides, is bisected at that point
- 9 In a parallelogram the perpendiculars drawn from one pair of opposite angles to the diagonal which joins the other pair are equal
 - 10 If ABCD is a parallelogram, and X, Y respectively the middle points of the sides AD, BC, shew that the figure AYCX is a paral lelogram

- 11 ABC and DEF are two triangles such that AB, BC are respectively equal to and parallel to DE, EF, show that AC is equal and parallel to DF
- 12 ABCD is a quadrilateral in which AB is parallel to DC, and AD equal but not parallel to BC, show that
 - (1) the $\angle A +$ the $\angle C = 180^{\circ} =$ the $\angle B +$ the $\angle D$,
 - (11) the diagonal AC=the diagonal BD.
- (iii) the quadrilateral is symmetrical about the straight line joining the middle points of AB and DC
- 13 AP, BQ are straight rods of equal length, turning at equal rates (both clockwise) about two fixed pivots A and B respectively If the rods start parallel but pointing in opposite senses, show that
 - (1) they will always be parallel,
- (n) the line joining PQ will always pass through a certain fixed point

(Miscellaneous Numerical and Graphical Examples)

14 Calculate the angles of the triangle ABC, having given

int.
$$\angle A = 3$$
 of ext $\angle A$, $3B = 4C$

- 15 A yield sailing due East changes her course successively by 63°, by 78°, by 119°, and by 64°, with a view to sailing round an island What further change must be made to set her once more on an Easterly course?
- 16 If the sum of the interior angles of a rectilineal figure is equal to the sum of the exterior angles, how many sides has it, and why?
- 17 Draw, using your protractor, any five sided figure ABCDE, in which

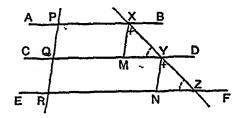
Verify by a construction with ruler and compasses that AE is parallel to BC, and account theoretically for this fact

- 18 A and B are two fixed points, and two straight lines AP, BQ, unlimited towards P and Q, are pivoted at A and B AP, starting from the direction AB, turns about A clockwise at the uniform rate of 7½° a second and BQ, starting simultaneously from the direction BA, turns about B counter clockwise at the rate of 3½° a second
 - (1) How many seconds will elapse before AP and BQ are parallel?
- (11) Find graphically and by calculation the angle between AP and BQ twelve seconds from the start
 - (iii) At what rate does this angle decrease?

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THEOREM 22

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal



Let the parallels AB, CD, EF cut off equal intercepts PQ, QR from the transversal PQR, and let XY, YZ be the corresponding intercepts cut off from any other transversal XYZ

It is required to prove that XY = YZ

Through X and Y let XM and YN be drawn parallel to PR

Proof Since CD and EF are parallel, and XZ meets them, the $\angle XYM =$ the corresponding $\angle YZN$

And since XM, YN are parallel, each being parallel to PR, the \angle MXY = the corresponding \angle NYZ

Now the figures PM, QN are parallelograms,

XM = the opp side PQ, and YN = the opp side QR,

and since by hypothesis PQ = QR,

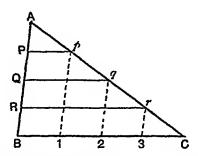
XM = YN

Then in the A' XMY, YNZ,

because
$$\begin{cases} \text{the } \angle XYM = \text{the } \angle YZN, \\ \text{the } \angle MXY = \text{the } \angle NYZ, \\ \text{and } XM = YN, \end{cases}$$

the triangles are identically equal, Theor 17 XY = YZ.

COROLLARY In a triangle ABC, if a set of lines Pp, Qq, Rr, , drawn parallel to the base, divide one side AB into equal parts, they also divide the other side AC into equal parts.



The lengths of the parallels Pp. Qq, Rr, , may thus be expressed in terms of the base BC

Through p, q, and r let p1, q2, r3 be drawn part to AB

Then by Theorem 22, these parts divide BC into four equal parts, of which Pp evidently contains one, Qq two, and Rr thic

In other words,

$$P_P = \frac{1}{4}$$
 BC, $Q_I = \frac{2}{4}$ BC $R_I = \frac{3}{4}$ BC

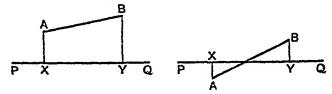
Similarly if the given paris divide AB into n equal parts

$$P_p = \frac{1}{n}$$
 BC, $Q_q = \frac{2}{n}$ BC, $R_t = \frac{3}{n}$ BC and so on

, Problem 7, p 78, should now be worked

DESTRICTION

If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the orthogonal projection of AB on PQ.



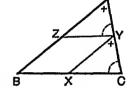
EXERCISES ON PARALLELS AND PARALLELOGRAMS

- 1 The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side.

[This is an important particular ease of Theorem 22

In the \triangle ABC, if Z is the middle point of AB, and ZY is drawn part to BC, we have to prove that AY=YC

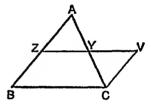
Draw YX par! to AB, and then prove the \triangle -ZAY, XYC congruent]



2 The straight line which joins the middle points of two sides of a triangle is parallel to the third side

[In the \triangle ABC, if Z, Y are the middle points of AB, AC, we have to prove ZY par' to BC

Produce ZY to V, making YV equal to ZY, and join CV! Prove the \triangle^n AYZ, CYV congruent, the rest follows at once]



- 3 The straight line which joins the middle points of two sides of a triangle is equal to half the third side
- 4 Show that the three straight lines which join the middle points of the sides of a triangle, divide it into four triangles which, are identically equal
- 5 Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.
- '6 ABCD is a parallelogram, and X, Y are the middle points of the opposite sides AD, BC shew that BX and DY trisect the diagonal AC
- 7 If the middle posits of adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram
- 8 Shew that the straight lines which join the middle mints of opposite sides of a quadrilateral, bisect one another

9 From two points A and B and from O the mid point between them perpendiculars AP, BQ, OX are drawn to a straight line CD If AP BQ measure respectively 42 cm and 58 cm, deduce the length of OX, and verify your result by measurement

Shew that OX=1(AP+BQ) or 1(AP-BQ), according as A and B are on the same side or on opposite sides of CD

- 10 When three parallels cut off equal intercepts from two trans versals show that of the three parallel lengths between the two transversals the middle one is the Arithmetic Mean of the other two
- The parallel sides of a trajezium are a centimetres and b centimetres in length. Prove that the line joining the middle points of the of tone endre se verrallel to the perrallel ender, and that its length is . 4(3+b) centimetre.
 - 12. OX and OY are two straight lines, and along OX five points 1, 2 3, 4, 5 are marked at equal distances. Through these points partilels are drawn in any direction to me t OY. Measure the lengths of these parallels take their average, and compare it with the length of the third parallel. Prove acome rically that the 3rd parallel is the mean of all five

State the corresponding theorem for any odd number (2n+1) of parallels so drawn

13 I rom the angular points of a parallelogram perpendiculars are drawn to any straight line which is outside the parallelogram shew that the sum of the perpendiculars drawn from one pair of opposite angles is equal to the sum or those drawn from the other pair

[Draw the diagonals and from the r point of intersection suppose a perpendicular drawn to the given straight I ne.]

- 14 The sum of the perpendiculars drawn from any point in the base of an isosciles trangle to the equal sides is equal to the perpen died a driver from either extremity of the bree to the opposite side
- Il. follows that the sum of the distances of any point in the base or an 150 celes trangle from the equal rides 15 constant, that is, the same whatever point in the base is taken]

How would this property be mounfied if the given point were taken in the base produced?

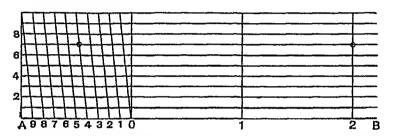
- The sum of the perpendiculars drawn from any point within an equilateral transle to the three sides is equal to the perpendicular drawn from any one of the angular points to the opposite side, and is therefore constant
- 16 Found and parallel lines have equal projections on any other strught line E

RSG

DIAGONAL SCALES

Diagonal scales form an important application of Theorem 22 We shall illustrate their construction and use by describing a Decimal Diagonal Scale to shew Inches, Tenths, and Hundredths

A straight line AB is divided (from A) into inches, and the points of division marked 0, 1, 2 The primary division 0A is subdivided into tenths, these secondary divisions being numbered (from 0) 1, 2, 3, 9 We may now read on AB inches and tenths of an inch



In order to read hundredths, ten lines are taken at any equal intervals parallel to AB, and perpendiculars are drawn through 0, 1, 2,

The primary (or inch) division corresponding to 0A on the tenth parallel is now subdivided into ten equal parts, and diagonal lines are drawn, as in the diagram, joining 0 to the first point of subdivision on the 10th parallel

The scale is now complete, and its use is shewn in the following example

Example To take from the scale a length of 2 47 inches

- (1) Place one point of the dividers at 2 in AB, and extend them till the other point reaches 4 in the subdivided inch 0A We have now 2.4 inches in the dividers
- (11) To get the remaining 7 hundredths, move the right-hand point up the perpendicular through 2 till it reaches the 7th parallel Then extend the dividers till the left point reaches the diagonal 4 also on the 7th parallel We have now 2 47 inches in the dividers.

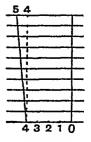
RIASON FOR THE ABOVE PROCESS

The first step needs no explanation The reason of the second is found in the Corollary of Theorem 22

Joining the point 4 to the corresponding point on the tenth parallel, we have a triangle 4,4,5, of which one side 4,4 is divided into ten equal parts by a set of lines parallel to the base 4,5

Therefore the lengths of the parallels between 4,4, and the diagonal 4,5 are $\frac{1}{10}$, $\frac{2}{10}$, $\frac{2}{10}$, of the base, which is 1 inch

Hence these lengths are respectively 01, 02, 03, of 1 mch



Similarly, by means of this scale, the length of a given straight line in it be measured to the nearest buildredth of an inch

Again, if one inch-division on the scale is taken to represent 10 feet, then 2 47 inches on the scale will represent 247 feet And it one inch-division on the scale represents 100 links, then 2 47 inches will represent 247 links. Thus a diagonal scale is of service in preparing plans of enclosures, buildings, or field works, where it is incressary that every dimension of the actual object must be represented by a line of proportional length on the plan

NOTE.

The subdivision of a diagonal scale need not be decimal For instance we might construct a diagonal scale to read centimetres, millimetres, and quarters of a millimetre, in which case we should take four parallels to the line AB

[I or Excresses on Linear Measurements see the following page]

EXERCISES ON LINEAR MEASUREMENTS

- Draw straight lines whose lengths are 1 25 inches, 2.72 inches, 3 08 inches
- 2 Draw a line 268 inches long, and measure its length in centi metres and the nearest millimetre
- 3 Draw a line 57 cm in length, and measure it in inches (to the nearest hundredth) Check your result by calculation, given that 1 cm = 0.3937 inch
- Find by measurement the equivalent of 3 15 inches in centimetres and millimetres Hence calculate (correct to two decimal places) the value of 1 cm in inches
- Draw lines 2.9 cm and 6.2 cm in length, and measure them in inches Use each equivalent to find the valve of 1 inch in centimetres and millimetres, and take the average of your results
- A distance of 100 miles is represented on a map by 1 inch Draw lines to represent distances of 336 miles and 408 miles
- 7 If 1 inch on a map represents I kilometre, draw lines to represent 850 metres, 2980 metres, and 1010 metres
- A plan is drawn to the scale of 1 inch to 100 links Measure in contimetres and millimetres a line representing 417 links
- Find to the nearest hundredth of an inch the length of a line which will represent 42 500 kilometres in a map drawn to the scale of 1 centimetre to 5 kilometres
- 10 The distance from London to Oxford (in a direct line) is 55 miles. If this distance is represented on a map by 2.75 inches, to what scale is the map drawn? That is, how many miles will be represented by 1 inch? How many kilometres by 1 centimetre?

 [1 cm = 0.3937 inch, 1 km. = 4 mile, nearly]
- On a map of France drawn to the scale 1 inch to 35 miles, the distance from Paris to Calais is represented by 42 inches Find the distance accurately in miles, and approximately in kilometres, and express the scale in metric measure [1 km = # mile, nearly]
- 12 The distance from Exeter to Plymonth is 37½ miles, and appears on a certain map to be 2½", and the distance from Lincoln to York is 88 km, and appears on another map to be 7 cm. Compare the scales of these maps in miles to the inch
- Draw a diagonal scale, 2 centimetres to represent 1 yard, shewing yards, feet, and inches.

ur Mathur

PRACTICAL GEOMETRY

PROBLEMS

The following problems are to be solved with ruler and compasses only No step requires the actual measurement of any line or angle, that is to say, the constructions are to be made without using either a graduated scale of length, or a protractor

The problems are not merely to be studied as propositions, but the construction in every case is to be actually performed by the learner, great care being given to accuracy of drawing

Each problem is followed by a theoretical proof, but the results of the work should always be verified by measurement, as a test of correct drawing Accurate measurement is also required in applications of the problems

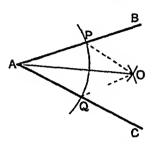
In the diagrams of the problems lines which are inserted only for purposes of proof are dotted, to distinguish them from lines necessary to the construction

For practical applications of the problems the student should be provided with the following instruments

- A flat ruler, one edge being graduated in centimeties and millimetres, and the other in inches and tenths
- Two set squares, one with angles of 45°, and the other with angles of 60° and 30°
 - A pair of pencil compasses 3
 - A pair of dividers, preferably with screw adjustment.
 - 5 A semi circular protractor

PROBLEM 1

To bisect a given angle



Lct BAC be the given angle to be bisected

Construction With centre A, and any radius, draw an arc of a circle cutting AB, AC at P and Q.

With centres P and Q, and radius PQ, draw two arcs cutting at O Join AO

Then the L BAC is bisected by AO

Proof

Join PO, QO

In the \triangle APO, AQO,

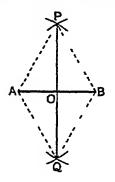
because $\begin{cases} AP = AQ, \text{ being radii of a circle,} \\ PO = QO, ,, , equal circles,} \\ \text{and AO is common,} \end{cases}$

the triangles are equal in all respects, Theor 7 so that the \angle PAO = the \angle QAO, that is, the \angle BAC is bisected by AO

Note centres P and Q, and the intersection of these arcs determines the point O Any radius, however, may be used instead of PQ, provided that it is great enough to secure the intersection of the arcs

PROBLEM 2

To bisect a given straight line



Let AB be the line to be bisected

Construction. With centre A, and radius AB, draw two arcs, one on each side of AB

With centre B, and radius BA, draw two arcs, one on each side of AB, cutting the first arcs at P and Q.

Join PQ, cutting AB at O Then AB is bisected at O

Proof.

Join AP, AQ, BP, BQ In the \triangle APQ, BPQ.

because AP = BP, being radii of equal circles,
AQ = BQ, for the same reason,
and PQ is common,

> the \angle APQ = the \angle BPQ. Ágain in the \triangle ² APO, BPO,

because $\begin{cases} AP = BP, \\ PO \text{ is common,} \\ and the <math>\angle APO = \text{the } \angle BPO, \end{cases}$

AO = OB, that is, AB is bisected at O

Theo 4

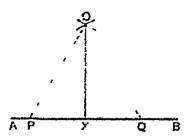
Theor 7°

Notes (1) AB was taken as the radius of the arcs drawn from the centres A and B, but any radius may be used provided that it is great enough to secure the intersection of the arcs which determine the points P and O.

(11) From the congruence of the \triangle *APO, BPO it follows that the \triangle AOP=the \triangle BOP. As these are adjacent angles, it follows that PQ bisects AB at right angles

/ Proplem 3

To draw a straight line perpendicular to a min a straight line at a given point in it



Let AB be the straight line, and X the point in it at which a perpendicular is to be drawn

Construction With centre X cut off from AB and two equal part YP, XQ.

With centres P and Q and radius PQ drive two ares cutting at O

Join XO Then XO is perpote AB

Proof

Join OP, OQ

In the . *OYP, OYQ

because (YP~YQ, In construction, OX is count on, and PO -QO, being ruly of equal circles

the LOXP - the LOXQ The r 'And these being adjacent angles, each is a right angle,

And these being adjacent angles, each is a right angle that is, XO is here to AB

Ohr If the point X is near one end of AB one or other of the alternative constructions on the next page should be used

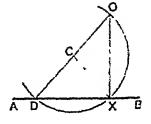
PROBLEM 3 SECOND METHOD

Construction. Take any point Coutside AB

With centre C, and radius CX, draw a circle cutting AB at D

Join DC, and produce it to meet the circumference of the circle at O Join XO

Then XO is perp to AB



Proof.

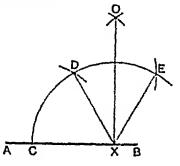
Because CO = CX, the \(\triangle CXO = \text{the } \(\triangle CXD = \text{the } \) \end{the } \(\triangle CXD = \text{the } \(\triangle CXD =

PROBLEM 3 THIRD METHOD

Construction With centre X and any radius, draw the arc CDE, cutting AB at C

With centre C, and with the same radius, draw an arc, cutting the first are at D

With centre D, and with the same radius, draw an arc cutting the first arc at E



I'rob 1

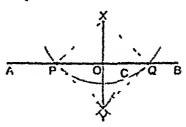
Bisect the LDXE by XO Then XO is perp to AB

Proof Each of the L*CXD, DXE may be proved to be 60° and the LDXO is half of the LDXE, the LCXO is 90°

That is, XO is perp to AB

PROPERTY 1

To draw a straight line perpendicular to a given straight line from a given external point



Let X be the given external point from which a perpendicular is to be drawn to AB

Construction Take any point C on the side of AB 16 mote from X

With centre X, and ridius XC, draw an are to cut AB at P and Q

With centres P and Q and rolms PX draw resenting at Y, on the side of AB opposite to X

Join XY cutting AB at O Then XO is perp to AB

Proof

Join PX QX PY, QY,

In the , Pay, QXY,

because PX = QX, being radii of a circle, PY = QY, for the same reison, and XY is common

the _PXY - flic _QXY

Theor T

Again, in the *PXO, QXO,

because $\begin{cases} PX = QX, \\ XO \text{ is common,} \\ and the <math>\angle PXO = \text{the } \angle QXO \end{cases}$

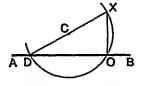
the LXOP = the LXOQ Theor 4

And these being adjacent angles, each is a right angle, that is, XO is perp to AB

Obs When the point X is nearly opposite one end of AB, one or other of the alternative constructions given below should be used.

PROBLEM 4 SECOND METHOD

Construction Take any point D in AB Join DX, and bisect it at C With centre C, and radius CX, draw a circle cutting AB at D and O Join XO



Then XO is perp to AB

For, as in Problem 3, Second Method, the \angle XOD is a right angle

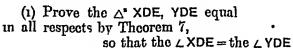
PROBLEM 4 THIRD METHOD

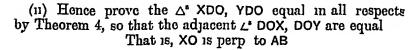
Construction Take any two points D and E in AB

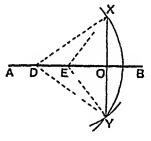
With centre D, and radius DX, draw an arc of a circle, on the side of AB opposite to X

With centre E, and radius EX, draw another are cutting the former at Y

Join XY, cutting AB at O Then XO is perp to AB

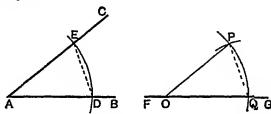






PROBLEM 5

At a given point in a given straight line to make an angle equal to a given angle



Let BAC be the given angle, and FG the given straight line, and let O be the point at which an angle is to be made equal to the \angle BAC

Construction With centre A, and with any radius, draw an arc cutting AB and AC at D and E

With centre O, and with the same radius, draw an are

cutting FG at Q.

With centre Q, and with radius DE, draw an are cutting the former arc at P

Join OP
Then POQ is the required angle.

Proof

Join ED, PQ.

In the \triangle POQ, EAD,

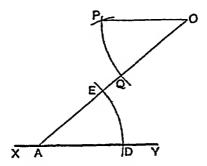
because $\begin{cases} \mathsf{OP} = \mathsf{AE}, \text{ being radu of equal circles,} \\ \mathsf{OQ} = \mathsf{AD}, \text{ for the same reason,} \\ \mathsf{PQ} = \mathsf{ED}, \text{ by construction,} \end{cases}$

the triangles are equal in all respects, so that the \angle POQ = the \angle EAD

Theor 7

PPOBLEM 6

Through a given point to draw a straight line parallel to a given straight line.



Let XY be the given straight line, and O the given point, through which a straight line is to be drawn part to XY

Construction. In XY take any point A, and join OA Using the construction of Problem 5, at the point O in the line AO make the \angle AOP equal to the \angle OAY and alternate to it

Then OP is parallel to XY

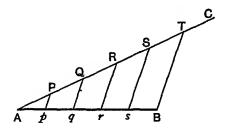
Proof Because AO, meeting the straight lines OP, XY, makes the alternate L*POA, OAY equal,

OP is par' to XY

** The constructions of Problems 3, 4, and 6 are not usually followed in practical applications Parallels and perpendiculars may be more quickly drawn by the aid of set squares (See LESSONS IN EXPERIMENTAL GEOMETRY, pp. 36, 42)



To divide a given straight line into any number of equal parts



Let AB be the given straight line, and suppose it is required to divide it into five equal parts

Construction. From A draw AC, a straight line of unlimited length, making any angle with AB

From AC mark off five equal parts of any length, AP, PQ, QR, RS, ST

Join TB, and through P, Q, R, S draw par to TB, meeting AB in p, q, r, s

Then since the parts Pp, Qq, Rr, Ss, TB cut off five equal parts from AT, they also cut off five equal parts from AB (Theorem 22)

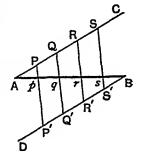
SECOND METHOD

From A draw AC at any angle with AB, and on it mark off four equal parts AP, PQ, QR, RS, of any length

From B draw BD par' to AC, and on it mark off BS', S'R', R'Q', Q'P', each equal to the parts marked on AC

Join PP, QQ', RR', SS' meeting AB in p, q, r, s Then AB is divided into five equal parts at these points

[Prove by Theorems 20 and 22]



EXERCISES ON LINES AND ANGLES

(Graphical Exercises)

- 1 Construct (with ruler and compasses only) an angle of 60° By repeated bisection divide this angle into four equal parts
- 2 By means of Exercise 1, trisect a right angle, that is, divide it into three equal parts

Bisect each part, and hence show how to trisect an angle of 45° [No construction is known for exactly trisecting any angle]

- 3 Draw a line 6.7 cm long, and divide it into five equal parts Measure one of the parts in inches (to the nearest hundredth), and verify your work by calculation [1 cm = 0.3937 inch]
- 4 From a straight line 372" long, cut off one seventh Measure the part in centimetres and the nearest millimetre, and verify your work by calculation
- 5 At a point X in a straight line AB draw XP perpendicular to AB, making XP 1 8" in length From P draw an oblique PQ, 3.0" long, to meet AB in Q Measure XQ.

(Problems State your construction, and give a theoretical proof)

6 In a straight line XY find a point which is equidistant from two given points A and B

When is this impossible?

7 In a straight line XY find a point which is equidistant from two intersecting lines AB, AC

When is this impossible?

- 8 From a given point P draw a straight line PQ, making with a given straight line AB an angle of given magnitude
- 9 From two given points P and Q on the same side of a straight hine AB, draw two lines which meet in AB and make equal angles with it.

[Construction From P draw PH perp to AB, and produce PH to P, making HP' equal to PH Join P'Q entting AB at K Join PK Prove that PK, QK are the required lines]

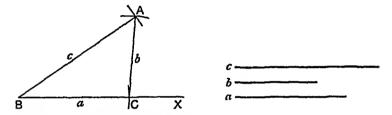
10 Through a given point P draw a straight line such that the perpendiculars drawn to it from two points A and B may be equal

Is this always possible?

THE CONSTRUCTION OF TRIANGLES

PROBLEM 8

To draw a triangle having given the lengths of the three sides



Let a, b, c be the lengths to which the sides of the required triangle are to be equal

Construction Draw any straight line BX, and cut off from it a part BC equal to a

With centre B, and radius c, draw an arc of a circle

With centre C, and radius b, draw a second arc cutting the first at A

Join AB, AC

Then ABC is the required triangle, for by construction the sides BC, CA, AB are equal to a, b, c respectively

Obs The three data, α , b, c may be understood in two ways either as three actual lines to which the sides of the triangle are to be equal, or as three numbers expressing the lengths of those lines in terms of inches, centimetres, or some other linear unit

Notes (1) In order that the construction may be possible it is necessary that any two of the given sides should be together greater than the third side (Theorem 11), for otherwise the arcs drawn from the centres B and C would not cut

(ii) The arcs which cut at A would, if continued, cut again on the other side of BC Thus the construction gives two triangles on opposite sides of a common base

ON THE CONSTRUCTION OF TRIANGLES

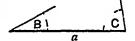
It has been seen (Page 50) that to prove two triangles identically equal, three parts of one must be given equal to the corresponding parts of the other (though any three parts do not necessarily serve the purpose) This amounts to saying that to determine the shape and size of a triangle we must know three of its parts or, in other words,

To construct a triangle three independent data are required

For example, we may construct a triangle

- (1) When two sides (b, c) and the included angle (A) are given. The method of construction in this case is obvious.
- (11) When two angles (A, B) and one side (a) are given Here, since A and B are given, we at once know C, for A+B+C=180°

Hence we have only to draw the base equal to a, and at its ends make angles equal to B and C, for we know that the remaining angle must necessarily be equal to A



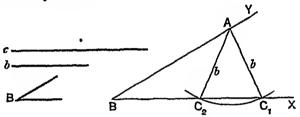
(111) If the three angles A, B, C are given (and no side), the problem is indeterminate, that is, the number of solutions is unlimited

For if at the ends of any base we make angles equal to B and C, the third angle is equal to A

This construction is indeterminate, because the three data are not independent, the third following necessarily from the other two

PROBLEM 9

To construct a triangle having given two sides and an angle opposite to one of them



Let b, c be the given sides and B the given angle

Construction Take any straight line BX, and at B make the AXBY equal to the given AB

From BY cut off BA equal to c

With centre A, and radius b, draw an arc of a circle

If this are cuts BX in two points C, and C, both on the same side of B, both the A*ABC, ABC, satisfy the given con ditions

This double solution is known as the Ambiguous Case, and will occur when b is less than c but greater than the perp from A on BX

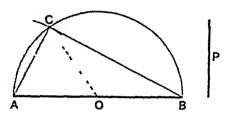
EXERCISE.

Draw figures to illustrate the nature and number of solutions in the following cases

- (1) When b is greater than c
- (11) When b is equal to c
- (iii) When b is equal to the perpendicular from A on BX (iv) When b is less than this perpendicular



To construct a right analed triangle having onen the hypotenuse and one sul-



Let AB be the hypotenuse and P the given side

Construction. Bisect AB at O, and with centre O, and radius OA, draw a semicircle

With centre A, and ridius P, driw an are to cut the semi-

Join AC BC Inen ABC is the required triangle

Proof

Join OC

Because OA = OC, the \(\text{OCA} = \text{the \(\text{LOAC} \)

And because OB = OC, the $\angle OCB =$ the $\angle OBC$

the whole
$$\angle ACB = \text{the } \angle OAC + \text{the } \angle OBC$$

= $\frac{1}{5}$ of 180° Thun 16
= $\frac{5}{1}$ 00°

ON THE CONSTRUCTION OF TRIANGLES

(Graphical Exercises)

- 1 Draw a triangle whose sides are 7.5 cm, 6.2 cm, and 5.3 cm Draw and measure the perpendiculars dropped on these sides from the opposite vertices
- [N B The perpendiculars, if correctly drawn, will meet at a point, as will be seen later See page 207]
- 2 Draw a triangle ABC, having given $\alpha=3~00$ °, b=2~50°, c=2~75° Bisect the angle A by a line which meets the base at X. Measure BX and XC (to the nearest hundredth of an inch), and hence calculate the value of $\frac{BX}{CX}$ to two places of decimals. Compare your result with

the value of $\frac{c}{b}$

- Two sides of a triangular field are 315 yards and 260 yards, and the included angle is known to be 39° Draw a plan (1 inch to 100 yards) and find by measurement the length of the remaining side of the field
- 4 ABC is a triangular plot of ground, of which the base BC is 75 metres, and the angles at B and C are 47° and 68° respectively. Draw a plun (scale 1 cm. to 10 metres). Write down without measurement the size of the angle A, and by measuring the plan, obtain the approximate lengths of the other sides of the field, also the perpendicular drawn from A to BC.
- 5 A yacht on leaving harbour steers N E sailing 9 knots an hour After 20 minutes she goes about, steering N W for 35 minutes and making the same average speed as before. How far is she now from the harbour, and what course (approximately) must she set for the run home? Obtain your results from a chart of the whole course, scale 2 cm to 1 knot.
- 6 Draw a right-angled triangle, given that the hypotenuse c=10 6 cm and one side a=5 6 cm Measure the third side b, and find the value of $\sqrt{c^2-a^2}$ Compare the two results
- 7 Construct a triangle, having given the following parts $B=34^{\circ}$, b=55 cm, c=85 cm Shew that there are two solutions Measure the two values of α , and also of C, and shew that the latter are supplementary
- 8 In a triangle ABC, the angle A=50°, and b=6.5 cm. Illustrate by figures the cases which arise in constructing the triangle, when (1) a=7 cm (11) a=6 cm (111) a=5 cm. (112) a=4 cm.

9 Two straight roads, which cross at right angles at A, are earried over a straight canal by bridges at B and C. The distance between the bridges is 461 yards, and the distance from the crossing A to the bridge B is 261 yards. Draw a plan, and by measurement of it ascertain the distance from A to C.

(Problems State your construction, and give a theoretical proof)

- 10 Draw an isosceles triangle on a base of 4 cm, and having an altitude of 6 2 cm. *Prove* the two sides equal, and measure them to the nearest millimetre
- 11 Draw an isosceles triangle having its vertical angle equal to a given angle, and the perpendicular from the vertex on the base equal to a given straight line

Hence draw an equilateral triangle in which the perpendicular from one vertex on the opposite side is 6 cm. Measure the length of a side to the nearest millimetre.

- 12 Construct a triangle ABC in which the perpendicular from A on BC is 50 cm, and the sides AB, AC are 58 cm and 90 cm respectively Measure BC
- 13 Construct a triangle ABC having the angles at B and C equal to two given angles L and M, and the perpendicular from A on BC equal to a given line P
- 14 Construct a triangle ABC (without protractor) having given two angles B and C and the side b
- 15 On a given base construct an isosceles triangle having its vertical angle equal to a given angle L.
- 16 Construct a right angled triangle, having given the length of the hypotennse c, and the sum of the remaining sides a and b

If c=5 3 cm, and a+b=7 3 cm, find a and b graphically, and calculate the value of $\sqrt{a^2+b^2}$

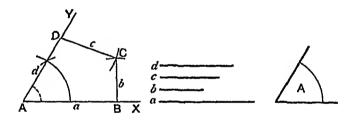
- /17 Construct a triangle having given the perimeter and the angles at the base For example, a+b+c=12 cm, $B=70^{\circ}$, $C=80^{\circ}$
- 18 Construct a triangle ABC from the following data a=6.5 em, b+c=10 cm, and $B=60^{\circ}$ Measure the lengths of b and c.
 - 19 Construct a triangle ABC from the following data a=7 cm, c-b=1 cm, and $B=55^{\circ}$ Measure the lengths of b and c

THE CONSTRUCTION OF QUADRILATERALS

It has been shewn that the shape and size of a triangle are completely determined when the lengths of its three sides are given. A quadrilateral, however, is not completely determined by the lengths of its four sides. From what follows it will appear that five independent data are required to construct a quadrilateral



To construct a quadrilateral, given the lengths of the four sides and one angle



Let a, b, c, d be the given lengths of the sides, and A the angle between the sides equal to a and d

Construction. Take any straight line AX, and cut off from it AB equal to a

Make the \angle BAY equal to the \angle A. From AY cut off AD equal to d

With centre D, and radius c, draw an arc of a circle
With centre B and radius b, draw another arc to cut the
former at C

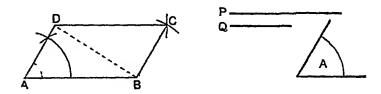
Join DC, BC

Then ABCD is the required quadrilateral, for by construction the sides are equal to a, b, c, d, and the \angle DAB is equal to the given angle

Them 7

PROBLEM 12

To construct a parallelogram having given two adjacent sides and the included angle



Let P and Q be the two given sides, and A the given angle

Construction 1 (With ruler and compasses) Take a line AB equal to P, and at A make the \angle BAD equal to the \angle A, and make AD equal to Q.

With centre D, and radius P, draw an are of a circle

With centre B, and radius Q, draw another are to cut the former at C

Then ABCD is the required parm

Proof.

Join DB.

In the A* DCB, BAD,

because $\begin{cases} DC = BA, \\ CB = AD, \\ and DB \text{ is common }, \end{cases}$

the $\angle CDB =$ the $\angle ABD$, and these are alternate angles,

DC is par' to AB

Also DC = AB,

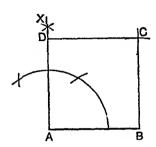
*. DA and BC are also equal and parallel Theor 20
ABCD is a par^m

Construction 2 (With set squares) Draw AB and AD as before, then with set squares through D draw DC par' to AB, and through B draw BC par' to AD

By construction, ABCD is a part having the required parts

PROBLEM 13

To construct a square on a given side



Let AB be the given side

Construction 1 (With ruler and compasses) At A draw AX perp to AB, and cut off from it AD equal to AB

With B and D as centres, and with radius AB, draw two arcs

cutting at C

Join BC, DC Then ABCD is the required square

Proof. As in Problem 12, ABCD may be shewn to be a pai^m And since the \(\perp BAD\) is a right angle, the figure is a rectangle Also, by construction all its sides are equal ABCD is a square

Construction 2 (With set squares) At A draw AX perp to AB, and cut off from it AD equal to AB

Through D draw DC part to AB, and through B draw BC

par' to AD meeting DC in C

Then, by construction, ABCD is a rectangle [Def 3, page 56] Also it has the two adjacent sides AB, AD equal it is a square.

EXERCISES

ON THE CONSTRUCTION OF QUADRILATERALS

1 Draw a rhombus each of whose sides is equal to a given straight line PQ, which is also to be one diagonal of the figure

Ascertain (without measurement) the number of degrees in each angle, giving a reason for your answer

- 2 Draw a squaro on a side of 25 inches Prove theoretically that its diagonals are equal, and by measuring the diagonals to the nearest numbered the of an inch test the correctness of your drawing
- 3 Construct a square on a diagonal of 3.0°, and measure the lengths of each side. Obtain the average of your results
- 4 Driw a parallelogram ABCD, having given that one side $AB=5.5\,\mathrm{cm}$, and the diagonals AC, BD are 8 cm, and 6 cm respectively Measure AD
- 5 The diagonals of a certain quadrilateral are equal, (each 6 0 cm.), and they bisect one another at an angle of 60° Shew that fire independent data are here given

Construct the quadrilateral Name its species, and give a formal proof of your answer Measure the perimeter. If the angle between the diagonals were increased to 90°, by how much per cent would the perimeter be increased?

6 In a quadrilateral ABCD,

AB=5.6 cm., BC=2.5 cm, CD=4.0 cm, and DA=3.3 cm. Shew that the shape of the quadrilateral is not settled by these data

Draw the quadrilateral when (1) A=30° (11) A=60° Why does the construction fail when A=100°°

Determine graphically the least value of A for which the construction fails

7 Shew how to construct a quadrilateral, having given the lengths of the four sides and of one diagonal. What conditions must hold among the data in order that the problem may be possible?

Illustrate your method by constructing a quadrilateral ABCD, when

- (1) AB=30", BC=17", CD=25", DA=28", and the diagonal BD=2.6" Measure AC
- (11) AB=36 cm , BC=77 cm , CD=68 cm , DA=51 em , and the diagonal AC=85 em Measure the angles at B and D

LOCI

DEFINITION The locus of a point is the path traced out by it when it moves in accordance with some given law

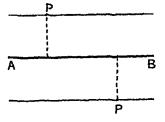
Example 1 Suppose the point P to move so that its distance from a fixed point O is constant (say I centimetre)

Then the locus of P is evidently the circum ference, of a circle whose centre is O and radius lem



Example 2 Suppose the point P moves at a constant distance (say 1 cm) from a fixed straight line AB

Then the locus of P is one or other of two straight lines parallel to AB, on either side, and at a distance of 1 cm from it

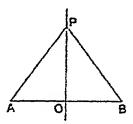


Thus the locus of a point, moving under some given condition, consists of the line or lines to which the point is thereby restricted, provided that the condition is satisfied by every point on such line or lines, and by no other

When we find a series of points which satisfy the given law, and through which therefore the moving point must pass, we are said to plot the locus of the point LOCI 91

PROPLEM 14

To find the locus of a point P which mores so that its distances from two fixed points A and B are always equal to one another



Here the point P moves through all positions in which PA = PB, one position of the moving point is at O the middle point of AB

Support P to be any other position of the moving point, that is, let PA = PB

Join OP

Then in the _ POA, POB,

because PO is common,
OA = OB,
and PA = PB, by hypothesis

the \angle POA = the \angle POB Hence PO is perpendicular to AB Theor 7

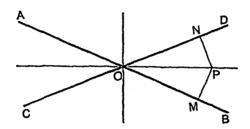
That is every point P which is equidistant from A and B lies on the straight line biseding AB at right angle.

Likewise it may be proved that every point on the perpendicular through O is equidistant from A and B

This line is therefore the required locus

PROBLEM 15

To find the locus of a point P which moves so that its perpendicular distances from two given straight lines AB, CD are equal to one another



Let P be any point such that the perp PM = the perp PN

Join P to O, the intersection of AB, CD

Then in the A' PMO, PNO,

because $\begin{cases} \text{the } \angle^{\bullet} \text{ PMO, PNO are } right \text{ angles,} \\ \text{the hypotenuse OP is common,} \\ \text{and one side PM = one side PN,} \end{cases}$

the triangles are equal in all respects, Theor 18 so that the $\angle POM = the \angle PON$

Hence, if P lies within the \angle BOD, it must be on the bisector of that angle,

and, if P is within the LAOD, it must be on the bisector of that angle

It follows that the required locus is the pair of lines which bisect the angles between AB and GD

INTERSECTION OF LOCI

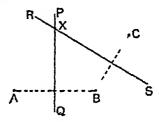
The method of Loci may be used to find the position of a point which is subject to two conditions. For corresponding to each condition there will be a locus on which the required point must be. Hence all points which are common to these two loci, that is, all the points of intersection of the loci, will satisfy loth the given conditions.

EXAMPLE 1 To find a point equidistant from three given points A, B, C, which are not in the same straight line

(1) The locus of points equidistant from A and B is the straight line PQ, which bisects AB at right angles

(ii) Similarly, the locus of points equidistant from B and C is the straight line RS which biscets BC at right angles

Hence the point common to PQ and RS must satisfy both conditions—that is to say, X the point of intersection of PQ and RS will be equidistant from A, B, and C



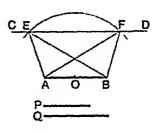
LEAMITY 2 To construct a triangle, having quen the base, the allitide and the length of the median which breets the base

Let AB be the given base, and P and Q the lengths of the altitude and median respectively

Then the triangle is known if its vertex is known

(i) Draw a straight line CD parallel to AB, and at a distance from it equal to P then the required vertex must be on CD

(ii) Again, from O the middle point of AB as centre, with radius equal to Q, de cribe a circle



then the required vertex must be on this circle.

Hence any points which are common to CD and the circle, satisfy both the given conditions—that is to say, if CD intersect the circle in E. F., each of the points of intersection might be the vertex of the required triangle—This supposes the length of the median Q to be greater than the altitude

It may happen that the data of the problem are so related to one another that the resulting loci do not intersect. In this case the problem is impossible

Obs In examples on the Intersection of Loci the student should make a point of investigating the relations which must exist among the data, in order that the problem may be possible, and he must observe that if under certain relations two solutions are possible, and under other relations no solution exists, there will always be some intermediate relation under which the two solutions combine in a single solution

EXAMPLES ON LOCI

- 1 Find the locus of a point which meves so that its distance (measured radially) from the circumference of a given circle is constant
- 2 A point P moves along a straight line RQ, find the position in which it is equidistant from two given points A and B
- 3 A and B are two fixed points within a circle find points on the circumference equidistant from A and B. How many such points are there?
- 4 A point P moves along a straight line RQ, find the position in which it is equidistant from two given straight lines AB and CD
- 5 A and B are two fixed points 6 cm apart. Find by the method of loci two points which are 4 cm distant from A, and 5 cm from B
- 6 AB and CD are two given straight lines Find points 3 cm. distant from AB, and 4 cm from CD How many solutions are there?
- 7 A straight rod of given length slides between two straight rulers placed at right angles to one another

Plot the locus of its middle point, and show that this locus is the fourth part of the circumference of a circle [See Problem 10]

- 8 On a given base as hypotenuse right-angled triangles are described Find the locus of thoir vertices
- 9 A is a fixed point, and the point X moves on a fixed straight line BC

Plot the locus of P, the middle point of AX, and prove the locus to be a straight line parallel to BC

 $10\,$ A is a fixed point, and the point X moves on the circumference of a given circle

Plot the locus of P, the middle point of AX, and prove that this locus is a circle [See Ex. 3, p 64.]

- 11 AB is a given straight line, and AX is the perpendicular drawn from A to any straight line passing through B If BX revolve about B, find the locus of the middle point of AX
- 12 Two straight lines OX, OY cut at right angles, and from P, a point within the angle XOY, perpendiculars PM, PN are drawn to OX, OY respectively—Plot the locus of P when
 - (1) PM PN is constant (=6 cm, gay)
 - (11) PM PN 14 const int (=3 cm , -27)

And in each case give a theoretical proof of the result you arrive at experimentally

13 Two strught lines OX, OY intersect at right angles at O, and from a novable point P perpendiculars PM PN are drawn to OX, OY

Plot (without proof) the locus of P, when

- (1) PM=2 PN,
- (11) PM=3 PN
- 14 Find a point which is at a given distance from a given point, and is equidistant from two given parallel straight lines

When does this problem admit of to colutions, when of one only, and when is a impossible?

- 15 S is a fixed point 2 inches distant from a given straight line MX. Find two points which are 27 inches distant from S, and also 27 inches distant from MX.
- 16 I and a series of points equidatant from a given point S and a given straight line MX Drive's curve freehand passing through all the points so found
- 17 On a gi en buse construct a triangle of given altitude, having its criex on a given straight line
 - 18 Find a point equidistant from the three sides of a triangle
- 19 Two strught lines OX, OY cut at right angles, and Q and R are points in OX and OY respectively. Plot the locus of the middle point of QR, when
 - (1) OQ + OR = constant
 - (u) OQ-OR=construt
- 29 S and S' are two fixed points. Find a series of noints P such that
 - (i) SP- SP=constant (siz 35 inches)
 - (u) SP-SP=constant (eag 1 5 meh)

In each case draw a curve freehand passing through all the points so found.

ON THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

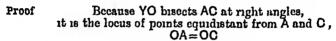
The perpendiculars drawn to the sides of a triangle from their middle points are concurrent

Let ABC be a A, and X, Y, Z the middle points of its sides

From Z and Y draw perps to AB, AC, meeting at O Join OX

It is required to prove that OX is perp to BC

Join OA, OB, OC



Agun, because ZO bisects AB at right angles. it is the locus of points equidistant from A and B. OA=OB

> Hence OB=OC O is on the locus of points equidistant from B and C that is, OX is perp to BC

Hence the perpendiculars from the mid points of the sides meet at O OED

П The bisectors of the angles of a triangle are concurrent

Let ABC be a △ Bisect the ∠ABC. BCA by straight lines which meet at O Join AO

It is required to prove that AO bisects the L BAC

From O draw OP, OQ, OR perp to the sides of the \(\Delta \)

Because BO bisects the LABC,

Proof it is the locus of points equidistant from BA and BC, OP=OR

Similarly CO is the locus of points equidistant from BC and CA, OP=OQ.

Hence OR=OQ. O is on the locus of points equidistant from AB and AC, that is, OA is the biscctor of the L BAC

QED

Hence the bisectors of the angles meet at O

III. The medians of a triangle are concurrent

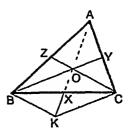
Let ABC be a \triangle Let BY and CZ be two of its medians, and let them intersect at O

Join AO, and produce it to meet BC in X

It is required to show that AX is the remaining median of the Δ

Through C draw CK parallel to BY, produce AX to meet CK at K

Join BK



Proof In the AKC, because Y is the middle point of AC, and YO is parallel to CK,
O is the middle point of AK

Theor 22.

Again in the ABK, since Z and O are the middle points of AB, AK, ZO is parallel to BK, that is, OC is parallel to BK the figure BKCO is a parm

But the diagonals of a parm bisect one another, X is the middle point of BC. That is, AX is a median of the \triangle

Hence the three medians meet at the point O QED

DEFINITION The point of intersection of the medians is called the centroid of the triangle

COROLLARY The three medians of a triangle cut one another at a point of trisection, the greater regment in each being towards the angular voint

For in the above figure it has been proved that AO=OK,

also that OX is half of OK,
OX is half of OA
that is, OX is one third of AX
Similarly OY is one third of BY,
and OZ is one third of CZ

Q E.D

By means of this Corollary it may be shewn that in any triangle the shorter median bisects the greater side

Note. It will be proved hereafter that the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent H B G

MISCELLANEOUS PROBLEMS

(A theoretical proof is to be given in each case)

- 1 A is a given point, and BC a given straight line From A draw a straight line to make with BC an angle equal to a given angle X How many such lines can be drawn?
- 2 Draw the bisector of an angle AOB, without using the vertex O in your construction.
- 3 P is a given point within the angle AOB Draw through P a straight line terminated by OA and OB, and bisected at P
- 4 OA, OB, OC are three straight lines meeting at O Draw a transversal terminated by OA and OC, and bisected by OB
- 5 Through a given point A draw a straight line so that the part intercepted between two given parallels may be of given length

When does this problem admit of two solutions? When of only one? And when is it impossible?

- 6 In a triangle ABC inscribe a rhombus having one of its angles coinciding with the angle A.
- 7 Use the properties of an equilateral triangle to trisect a given straight line

(Construction of Triangles)

- 8 Construct a triangle, having given
 - (1) The middle points of the three sides
- (11) The lengths of two sides and of the median which bisects the third side
- (111) The lengths of one side and the medians which bisect the other two sides
- (IV) The lengths of the three medians

PART II

ON AREAS

DEFINITIONS

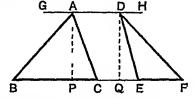
- 1 The altitude (or height) of a parallelogram with reference to a given side as base, is the perpendicular distance between the base and the opposite side
- 2 The altitude (or height) of a triangle with reference to a given side as base, is the perpendicular distance of the opposite vertex from the base

Note. It is clear that parallelograms or triangles which are between the same parallels have the same altitude

For let AP and DQ be the altatudes of the \triangle ABC, DEF, which are between the same parallels BF, GH

Then the fig APQD is evidently a rectangle,

AP=DQ



- 3 The area of a figure is the amount of surface contained within its bounding lines
- 4 A square inch is the area of a square drawn on a side one inch in length

Square inch

5 Similarly a square centimetre is the area of a square drawn on a side one centimetre in length

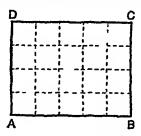
Sq

The terms square yard, square foot, square metre are to be understood in the same sense

6 Thus the unit of area is the area of a square on a side of unit length

THEOREM 23

Area of a rectangle If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area



Let ABCD represent a rectangle whose length AB is 5 feet, and whose breadth AD is 4 feet

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division of each line draw parallels to the other

The rectangle ABCD is now divided into compartments, each of which represents one square foot

Now there are 4 rows, each containing 5 squares, the rectangle contains 5×4 square feet

Similarly, if the length = a linear units, and the breadth = blinear units

the rectangle contains ab units of area.

And if each side of a square = a linear units, the square contains a2 units of area

These statements may be thus abridged-

the area of a rectangle = length × breadth (1),
the area of a square =
$$(side)^2$$
 (11)

OED

(1) Rectangles which have equal lengths and COROLLARIES equal breadths have equal areas

(11) Rectangles which have equal areas and equal lengths have also equal breadths

NOTATION

The rectangle ABCD is said to be contained by AB, AD, for these adjacent sides fix its size and shape

A rectangle whose adjacent sides are AB, AD is denoted by red AB, AD, or simply AB \times AD

A square drawn on the side AB is denoted by sq on AB, or AB2

EXERCISES

(On Tables of Length and Area)

- 1 Draw a figure to show irhy
 - (1) 1 m yard = 32 sq feet
 - (n) I sq foot =12- sq inches
 - (m) 1 eq cm = 10° eq mm
- 2. Draw a figure to show that the square on a straight line is four times the square on bulf the line
- 3 Use squared paper to show that the square on $1''=10^2$ times the square on 0.1''
- 4 If 1" represents 5 miles, what does an area of 6 square inches represent?

EXTENSION OF THEOREM 23

The proof of Theorem 23 here given supposes that the length and breadth of the given rectangle are expressed by whole numbers, but the formula holds good when the length and breadth are fractional

This may be illustrated thus

Suppose the length and breadth are 3.2 cm and 2.4 cm, we shall show that the area is (3.2×2.1) sq. cm

For length = 32 cm = 32 mm
breadth = 24 cm = 24 mm
area = (32 × 24) sq mm =
$$\frac{32 \times 24}{10^2}$$
 sq cm
= (32 × 24) sq cm

EXERCISES

(On the Area of a Rectangle.)

Draw on squared paper the rectangles of which the length (a) and breadth (b) are given below Calculate the areas, and verify by the actual counting of squares

1	a=2", b=3"	2	a=15'', b=4''
3	a=0.8'', $b=3.5''$	4	a=25, $b=14$ "
5	$a=2\cdot 2'', b=1\cdot 5''$	6	$a=16^{\circ}, b=21^{\circ}$

Calculate the areas of the rectangles in which

7 a=18 metres, b=11 metres. 8 a=7 ft b=72 m. 9 a=2.5 km, b=4 metres. 10 $a=\frac{1}{2}$ mile, b=1 inch

11 The area of a rectangle is 30 sq cm, and its length is 6 cm Find the breadth Draw the rectangle on squared paper, and verify your work by counting the squares

- 12 Find the length of a rectangle whose area is 39 sq in, and breadth 15" Draw the rectangle on squared paper, and verify your work by counting the squares
- 13 (1) When you treble the length of a rectangle without altering its breadth, how many times do you multiply the area?
- (11) When you treble both length and breadth, how many times do you multiply the area?

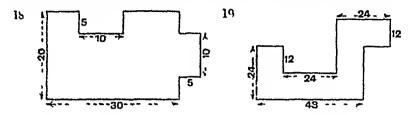
Draw a figure to illustrate your answers, and state a general rule

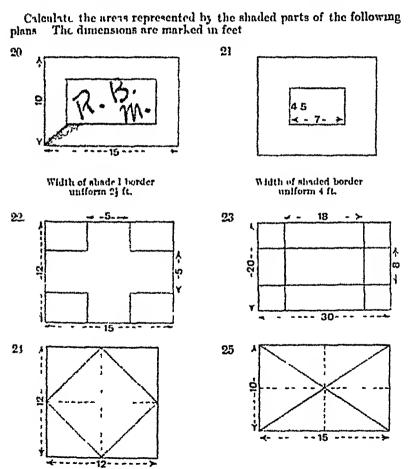
14 In a plan of a rectangular garden the length and breadth are 3 6" and 2 5', one much standing for 10 yards Find the area of the garden

If the arca is increased by 300 sq yds, the breadth remaining the same, what will the new length be? And how many inches will represent it on your plan?

- 15 Find the area of a rectangular enclosure of which a plan (scale 1 cm to 20 metres) incasures 6.5 cm by 4.5 cm
- 16 The area of a rectangle is 1440 sq yds. If in a plan the sides of the rectangle are 32 cm and 45 cm, on what scale is the plan drawn?
- 17 The area of a rectangular field is 52000 sq ft On a plan of this, drawn to the scale of 1" to 100 ft, the length is 3.25" What is the breadth?

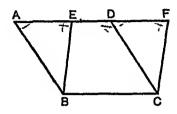
Calculate the areas of the enclosures of which plans are given below All the angles are right angles, and the dimensions are marked in feet





THEOREM 24 [Euclid I 35]

Parallelograms on the same base and between the same parallels are equal in area



Let the par^{ma} ABCD, EBCF be on the same base BC, and between the same par^{ta} BC, AF

It is required to prove that

the parm ABCD = the parm EBCF in area

Proof

In the A' FDC, EAB,

because the ext \angle FDC = the int opp \angle EAB, Theor 14
the int. \angle DFC = the ext \angle AEB,

the \triangle FDC = the \triangle EAB

Theor 17

Now, if from the whole fig ABCF the \triangle FDC is taken, the remainder is the par ABCD

And if from the whole fig ABCF the \triangle EAB is taken, the remainder is the par^m EBCF

these remainders are equal,

that is, the par^m ABCD = the par^m EBCF Q E D

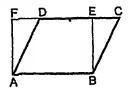
EXERCISE

In the above diagram the sides AD, EF overlap Draw diagrams in which (1) these sides do not overlap, (11) the ends E and D coincide

Go through the proof with these diagrams, and ascertain if it applies to them without change.

THE AREA OF A PARALLELOGRAM.

Let ABCD be a parallelogram, and ABEF the rectangle on the same base AB and of the same altitude BE Then by Theorem 24,



area of par^{an} ABCD = area of rect ABEF = AB × BE

= AB × BE = base × altitude

COROLLARY Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal

ın area

EXERCISES

(Numerical and Graphical)

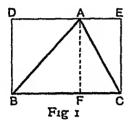
- 1 Find the area of parallelograms in which
 - (1) the base=5 5 cm, and the height=4 cm
 - (11) the base=24'', and the height=15''
- 2 Draw a parallelogram ABCD having given $AB=2\frac{1}{2}$, $AD=1\frac{1}{2}$, and the $\angle A=65^{\circ}$ Draw and measure the perpendicular from D on AB, and hence calculate the approximate area Why approximate?

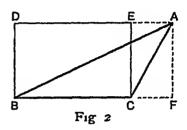
Again calculate the area from the length of AD and the perpendicular on it from B Obtain the average of the two results

- 3 Two adjacent sides of a parallelogram are 30 metres and 25 metres, and the included angle is 50° Draw a plan, 1 cm representing 5 metres, and by measuring each altitude, make two independent calculations of the area Give the average result
- 4. The area of a parallelogram ABCD is 42 sq in, and the base AB is 28" Find the height. If AD=2", draw the parallelogram
- 5 Each side of a rhombus is 2^n , and its area is 3.86 sq in Calculate an altitude. Hence draw the rhombus, and measure one of its acute angles

THEOREM 25

The Area of a Triangle The area of a triangle is half the area of the rectangle on the same base and having the same altitude





Let ABC be a triangle, and BDEC a rectangle on the same base BC and with the same altitude AF

It is required to prove that the ABC is half the rectangle BDEC

Proof Since AF is perp to BC, each of the figures DF, EF is a rectangle

Because the diagonal AB bisects the rectangle DF, the \triangle ABF is half the rectangle DF

Similarly, the AFC is half the rectangle FE adding these results in Fig. 1, and taking the difference in Fig. 2,

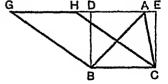
the ABC is half the rectangle BDEC

QED

TCOROLLARY A triungle is half any parallelogram on the same pase any between the same parallels

For t e \(\Delta \) ABC is half the rect BCED
And the rect BCED = any par BCHG
in the same base and between the same
par 16

the ABC is half the parm BCHG



AREAS 107

THE ARFA OF A TRIANGLE

If BC and AF respectively contain a units and p units of length, the rectangle BDEC contains ap units of area

the area of the $_ABC = \frac{1}{2}ap$ units of area

This result may be stated thus

Area of a Triangle = 1 base x altitude

EXFRCISES ON THE ARFA OF A TRIANGLE.

(Numerical and Graphical)

- 1 Calculate the areas of the trangles in which
 - (1) the base = 24 ft, the height = 15 ft
 - (11) the base = 45° , the height = 35°
 - (m) the base = 160 metres, the height = 125 metres
- 2 Draw triangles from the following data. In each case draw and measure the altitude with reference to a given side as base, hence calculate the approximate area.
 - (1) a=8.4 cm, b=6.8 cm, r=4.0 cm
 - (n) b=50 cm, c=08 cm, $A=65^{\circ}$
 - (iii) a = 6.5 cm, $B = 52^\circ$, C = 76
- /3 ABC is a triangle right angled at C show that its area = $\frac{1}{2}$ BC \sim CA Given a=6 cm b=5 cm, calculate the area

Drive the triangle and measure the hypotenuse c, draw and measure he perpendicular from C on the hypotenuse hence calculate the pproximate area

Note the error in your approximate result, and express it as a per entage of the true value

- 4 Repeat the whole process of the last question for a right-angled rangle ABC, in which $a=2.9^{\circ}$ and $b=4.5^{\circ}$, C being the right angle s before
 - 5 In a triangle, given
 - (1) Area=80 sq in, bise=1 ft 8 in, calculate the altitude
 - (ii) Area=104 sq cm, altitude=16 cm, calculate the base
- 6 Construct a triangle ABC, having given $a=30^\circ$, $b=28^\circ$, $c=26^\circ$ lray and measure the perpendicular from A on BC, hence calculate he approximate area

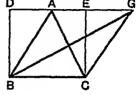
THEOREM 26 [Euclid I 37]

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area.

D
A
E
G

Let the \triangle^s ABC, GBC be on the same base BC and between the same parts BC, AG

It is required to more that the \triangle ABC = the \triangle GBC in area



Proof If BCED is the lectangle on the base BC, and between the same parallels as the given triangles, the \triangle ABC is half the rect BCED, Theor 25

also the \triangle GBC is half the rect BCED, the \triangle ABC = the \triangle GBC

QED

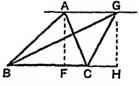
Similarly, triangles on equal bases and of equal altitudes are equal in area

THEOREM 27 [Euclid I 39]

If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels

Let the \triangle ABC, GBC, standing on the same base BC, be equal in area, and let AF and GH be their altitudes

It is required to prove that AG and BC are par



Proof The \triangle ABC is half the rectangle contained by BC and AF,

and the \triangle GBC is half the reetangle contained by BC and GH,

the rect BC, AF = the rect BC, GH,

AF = GH

Theor 23, Cor 2

Also AF and GH are par¹,
hence AG and FH, that is BC, are par¹

QED

EXERCISES ON THE AREA OF A TRIANGLE

(Theoretical)

- 1 ABC is a triangle and XY is drawn parallel to the base BC, cutting the other sides at X and Y Join BY and CX, and shew that
 - (1) the $\triangle XBC = the \triangle YBC$,
 - (11) the & BXY=the & CXY, Y
 - (iii) the $\triangle ABY = the \triangle ACX$

If BY and CX cut at K, shew that

h

(iv) the ∠BKX=the △CKY

3/2. Shew that a median of a triangle divides it into two parts of equal area.

How would you divide a triangle into three equal parts by straight lines drawn from its vertex?

- 3 Prove that a parallelogram is divided by its diagonals into four imangles of equal area
- 1/4 ABC is a triangle whose base BC is bisected at X. If Y is any point in the median AX, show that

the \angle ABY=the \triangle ACY in area

5 ABCD is a parallelogram, and BP, DQ are the perpendiculars from B and D on the diagonal AC

Show that BP=DQ

Hence if X is any point in AC, or AC produced,

prove (1) the $\triangle ADX =$ the $\angle ABX$,

(11) the ∠CDX=the ∠CBX

- I so some by means of Theorems 26 and 27 that the straight line soming the middle points of two sides of a triangle is parallel to the third pide.
- 77 The straight line which joins the middle points of the oblique sides of a trapezium is parallel to each of the parallel sides
- 8 ABCD is a parallelogram and X, Y are the middle points of the sides AD, BC, if Z is any point in XY, or XY produced, shew that the triangle AZB is one quarter of the parallelogram ABCD
- 9 If ABCD is a parallelogram, and X, Y any points in DC and AD respectively shew that the triangles AXB, BYC are equal in area
- 10 ABCD is a parallelogram, and P is any point within it, shew that the sum of the triangles PAB, PCD is equal to half the parallelogram

EXERCISES ON THE AREA OF A TRIANGLE

(Numerical and Graphical)

- 1 The sides of a triangular field are 370 yds, 200 yds, and 190 yds Draw a plan (scale 1" to 100 yards) Draw and measure an altitude, hence calculate the approximate area of the field in square yards
- 2 Two sides of a triangular enclesure are 124 metres and 144 metres respectively, and the included angle is observed to be 45° Draw a plan (scale 1 cm to 20 metres) Make any necessary measure ment, and calculate the approximate area
- 3 In a triangle ABC, given that the area = 6.6 sq cm, and the base BC = 5.5 em, find the altitude Hence determine the locus of the vertex A.

If in addition to the above data, BA=26 em, construct the triangle, and measure CA

- 4 In a triangle ABC, given arca=3.06 sq in , and $a=3.0^{\circ}$ Find the altitude, and the locus of A Given $C=68^{\circ}$, construct the triangle, and measure b
- 5 ABC is a triangle in which BC, BA have constant lengths 6 cm and 5 cm. If BC is fixed, and BA revolves about B, trace the changes in the area of the triangle as the angle B increases from 0° to 180°.

Answer this question by drawing a series of triangles, increasing B by increments of 30° Find the area in each case and tabulate the results

(Theoretical)

- 6 If two triangles have two sides of one respectively equal to two sides of the other, and the angles contained by those sides apple mentary, show that the triangles are equal in area Can such triangles ever be identically equal?
- 7 Shew how to draw on the base of a given triangle an isoseeles triangle of equal area.
- 8 If the middle points of the sides of a quadrilateral are joined in order, prove that the *parallelogram* so formed [see Ex 7, p 64], is half the quadrilateral
- 9 ABC is a triangle, and R, Q the middle points of the sides AB, AC, show that if BQ and CR intersect in X, the triangle BXC is equal to the quadrilateral AQXR
- 10 Two triangles of equal area stand on the same base but on coppesite sides of it show that the straight line joining their vertices is bisected by the base, or by the base produced

[The method given below may be omitted from a first course any case it must be postponed till Theorem 29 has been rend] Ιn

The Area of a Triangle Giren the three sides of a triangle, \vee_{to} calculate the area

Examire. Find the area of a triangle whose sides measure 21 m., 17 m, and 10 m

Let ABC represent the given

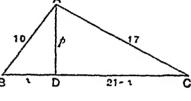
trangle.

Driv AD perp to BC, and denote AD by p

We shall first find the length of BD

Let BD=x metres then DC

=21-x metres From the right angled ADB, we have by Theorem 29



$$AD^2 = AB^2 - BD = 10^2 - x^2$$

And from the right angled A ADC,

AD²=AC ~ DC²=17-(21-x)ⁿ,

$$10-x=17^2-(21-x)^n$$

 $100-x^2=289-441+42x-x^2$.

OF whence

or

- x=6

Again,

AD"=AB - BD". $n^2 = 10 - 6^2 = 64$. n=S

Now Area of 'rrangle=\frac{1}{2} base \times altitude $=(4 \times 21 \ \text{S}) \text{ so } m = 84 \text{ so } m$

1 \FRCISES

Find by the above method the area of the triangles, whose sides are as follows

20 ft, 13 ft, 11 ft

- 15 yds , 14 yds , 13 yds
- 21 m, 20 m, 13 m
- 4 30 em., 25 cm, 11 em
- 37 ft, 30 ft, 13 ft

- 51 m, 37 m, 20 m 6
- If the given sider are a, b and c units in length, prove

(i)
$$x = \frac{a^2 + c^2 - b^2}{2a}$$
, (ii) $y^2 = c^2 - \left\{\frac{a^2 + c^2 - b^2}{2a}\right\}^2$,

(111) $\triangle = \frac{1}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}$

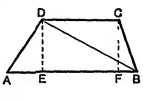
THE AREA OF QUADRILATERALS

THEOREM 28

10 find the area of

- (1) a trapezium
- (11) any quadrilateral
- (1) Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB

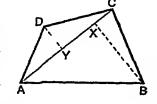
Let the parallel sides AB, CD measure a and b units of length, and let the A height CF contain b units



Then the area of ABCD = \triangle ABD + \triangle DBC = $\frac{1}{2}$ AB DE + $\frac{1}{2}$ DC CF = $\frac{1}{9}ah + \frac{1}{9}bh = \frac{h}{9}(a+b)$

the area of a trapezium = $\frac{1}{2}$ height × (the sum of the parallel sides)

(11) Let ABCD be any quadrilateral Draw a diagonal AC, and from B and D draw perpendiculars BX, DY to AC These perpendiculars are called offsets



If AC contains d units of length, and BX, DY p and q units respectively,

the area of the quad¹ ABCD =
$$\triangle$$
 ABC + \triangle ADC
= $\frac{1}{2}$ AC BX + $\frac{1}{2}$ AC DY
= $\frac{1}{2}dp + \frac{1}{2}dq = \frac{1}{2}d(p+q)$

That is to say,

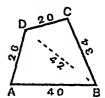
the area of a quadrilateral $=\frac{1}{2}$ diagonal \times (sum of offsets).

EXERCISES

(Numerical and Graphical)

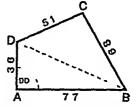
- 1 Find the area of the trapezium in which the two parallel sides are 4 7" and 3 3", and the height 1 5"
- 2 In a quadrilateral ABCD, the diagonal AC=17 feet, and the offsets from it to B and D are 11 feet and 9 feet. Find the area
- 3 In a plan ABCD of a quadrilateral enclosure, the diagonal AC measures 8-2 cm, and the offsets from it to B and D are 3 1 cm and 2-6 cm respectively. If 1 cm in the plan represents 5 metres, find the area of the enclosure
- 4 Draw a quadrilateral ABCD from the adjoining rough plan, the dimensions heing given in inches

Draw and measure the offsets to A and C from the diagonal BD and hence calculate the area of the quadrilateral



5 Draw a quadrilateral ABCD from the details given in the adjoining plan. The dimensions are to be in centimetres.

Make any necessary measurements of your figure, and calculate its area



- 6 Draw a trapezium ABCD from the following data AB and CD are the parallel sides AB=1", AD=BC=2", the ∠A=the ∠B=60° Make any necessary measurements, and calculate the area
- 7 Draw a trapezium ABCD in which AB and CD are the parallel sides, and AB=9 cm, CD=3 cm, and AD=BC=5 cm

Make any necessary measurement, and calculate the area

8 From the formula area of quad'=2 diag v (sum of offsets) show that, if the diagonals are at right angles,

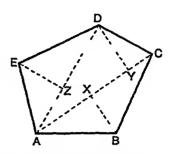
area = 1 (product of diagonals)

9 Given the lengths of the diagonals of a quadrilateral, and the angle between them, prove that the area is the same wherever they intersect.

THE AREA OF ANY RECTILINEAL FIGURE

1st METHOD A rectilineal figure may be divided into triangles whose areas can be separately calculated from suitable measurements. The sum of these areas will be the area of the given figure

Example. The measurements required to find the area of the figure ABCDE are AC, AD, and the offsets BX, DY, EZ.

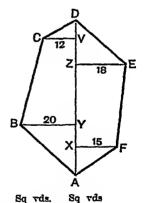


2nd METHOD The area of a rectlineal figure is also found by taking a base-line (AD in the diagram below) and offsets from it These divide the figure into right-angled triangles and right-angled trapeziums, whose areas may be found after measuring the offsets and the various sections of the base-line

Example. Find the area of the enclosure ABCDEF from the plan and measurements tabulated below

V=ŏ0 \Z=40	ZE=18
Y=18	
X=10	XF=15

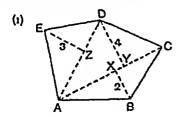
The measurements are made from A along the base line to the points from which the offsets spring



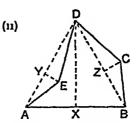
Here \triangle AXF= $\frac{1}{2}$ AX × XF = $\frac{1}{2}$ × 10 × 15 = 75 \triangle AYB= $\frac{1}{2}$ AY × YB = $\frac{1}{2}$ × 18 × 20 = 180 \triangle DZE= $\frac{1}{2}$ DZ × ZE = $\frac{1}{2}$ × 16 × 18 = 144 \triangle DVC= $\frac{1}{2}$ DV × VC = $\frac{1}{2}$ × 6 × 12 = 36 trap^m XFEZ= $\frac{1}{2}$ XZ × (XF+ZE)= $\frac{1}{2}$ × 30 × 33 = 495 trap^m YBCV= $\frac{1}{2}$ YV × (YB+VC)= $\frac{1}{2}$ × 32 × 32 = 512 , by addition, the fig ABODEF = 1442 sq yds

FYFRCISES

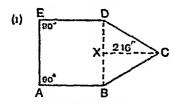
1 Calculate the areas of the figures (1) and (11) from the plans and dimensions (in cms.) given below



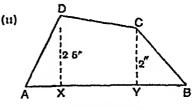
AC=6 cm, AD=5 cm Lengths of offsets figured in diagram



2. Draw full size the figures whose plans and dimensions are given below, and calculate the area in each case

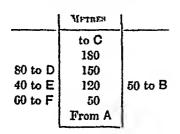


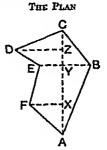
The fig is equilateral each side to be 24"



$$AX = 1\frac{1}{4}$$
", $XY = 2\frac{3}{4}$ ", $YB = 1\frac{1}{4}$ "

3 Find the area of the figure ABCDEF from the following measure ments and draw a plan in which 1 cm represents 20 metres





EXERCISES ON QUADRILATERALS

(Theoretical)

- 1 ABCD is a rectangle, and PQRS the figure formed by joining in order the middle points of the sides
 - Prove (1) that PQRS is a rhombus,
 - (11) that the area of PQRS is half that of ABCD

Hence shew that the area of a rhombus is half the product of its diagonals

Is this true of any quadrilateral whose diagonals cut at right angles? Illustrate your answer by a diagram

2. Prove that a parallelogram is bisected by any straight line which passes through the middle point of one of its diagonals

Hence shew how a parallelogram ABCD may be bisected by a straight line drawn

- (1) through a given point P,
- (11) perpendicular to the side AB,
- (111) parallel to a given line QR

Join the trapezium ABCD, AB is pirallel to DC, and X is the middle point of BC. Through X draw PQ parallel to AD to meet AB and DC produced at P and Q. Then prove

- (i) trapezium ABCD=parm APQD
- (11) trapezium ABCD=twice the △AXD

(Graphical)

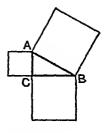
4 The diagonals of a quadrilateral ABCD ent at right angles, and measure 3 0" and 2 2" respectively Find the area

Shew by a figure that the area is the same wherever the diagonals cut, so long as they are at right angles.

- 5 In the parallelogram ABCD, AB=80 cm, AD=32 cm, and the perpendicular distance between AB and DC=30 cm Draw the par allelogram Calculate the distance between AD and BC, and check your result by measurement
- 6 One side of a parallelogram is 25", and its diagonals are 34" and 24" Construct the parallelogram, and, after making any necessary measurement, calculate the area
- 7 ABCD is a parallelogram on a fixed base AB and of constant area Find the locus of the intersection of its diagonals

EXERCISES LEADING TO THEOREM 29

In the adjoining diagram, ABC is a triaugle right angled at C, and squares are drawn on the three sides. Let us compare the area of the squares on the hypotennse AB with the sum of the squares on the sides AC, CB which coutain the right angle



1 Draw the above diagram, making AC=3 cm, and BC=4 cm;

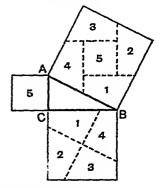
Then the area of the square on $AC = 3^2$, or 9 sq cm. and the square on $BC = 4^2$, or 16 sq cm.

the snm of the squares on AC, BC= 25 sq cm

Now measure AB, hence calculate the area of the square on AB, and compare the result with the sum already obtained

- 2. Repeat the process of the last exercise, making AC=1 0", and BC=2 4"
- 3 If a=15, b=8, c=17, shew arithmetically that $c^2=a^2+b^2$ Now draw on squared paper a triangle ABC, whose sides a, b, and c are 15, 8, and 17 units of length, and measure the angle ACB
- 4 Take any triangle ABC, rightangled at C, and draw squares on AC, CB, and on the hypotenuse AB

Through the mid-point of the square on CB (i.e. the intersection of the diagonals) draw lines parallel and perpendicular to the hypotenuse, thus dividing the square into four congruent quadrilaterals. These together with the square on AC, will be found exactly to fit into the square on AB, in the way indicated by corresponding numbers



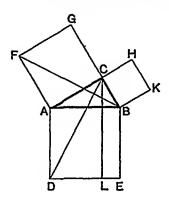
These experiments point to the conclusion that

In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides

A formal proof of this theorem is given on the next page

THEOREM 29 [Euclid I 47]

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides



Let ABC be a right-angled \triangle , having the angle ACB a rt \angle

It is required to prove that the square on the hypotenuse AB = the sum of the squares on AC, CB

On AB describe the sq ADEB, and on AC, CB describe the sqq ACGF, CBKH

Through C draw CL par to AD or BE Join CD, FB

Proof Because each of the La ACB, ACG is a rt L, BC and CG are in the same st line

Now the rt \angle BAD = the rt \angle FAC, add to each the \angle CAB then the whole \angle CAD = the whole \angle FAB

Then in the \triangle CAD, FAB,

because $\begin{cases} CA = FA, \\ AD = AB, \\ and the included <math>\angle CAD = the included \angle FAB, \\ the \triangle CAD = the \triangle FAB \end{cases}$ Theor 4

Now the rect. AL is double of the \triangle CAD, being on the same base AD, and between the same part AD, CL.

And the sq GA is double of the \triangle FAB, being on the same base FA, and between the same parts FA, GB

the rect AL=the sq GA

Similarly by joining CE, AK, it can be shewn that the reet BL=the sq HB

the whole sq AE = the sum of the sqq GA, HB that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB

QED

Obs This is known as the Theorem of Pythagoras The result established may be stated as follows

$$AB^2 = BC^2 + CA^2$$

That is, if a and b denote the lengths of the sides containing the right angle, and if c denotes the hypotenuse,

 $c^2 = a^2 + b^2$

Hence

 $a^2 = c^2 - b^2$, and $b^2 = c^2 - a^2$

Note 1 The following important results should be noticed

If CL and AB intersect in O, it has been shewn in the course of the
proof that

the sq GA=the reet AL, that 14, AC=the rect contained by AB AO (1)

Also the sq HB=the rect BL, that 19, BC'=the rect contained by BA, BO (11)

Note 2 It can be proved by superposition that squares standing on equal sides are equal in area

Hence we conclude, conversely,

If two squares are equal in area they stand on equal sides

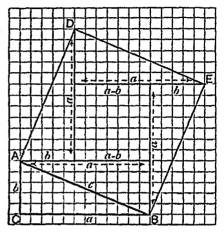
EXPERIMENTAL PROOFS OF PYTHAGORAS'S THEORFM.

I Here ABC is the given rt angled \triangle , and ABED is the square on the hypotenuse AB

By drawing lines par' to the sides BC, CA, it is easily seen that the sq BD is divided into 4 rt angled Δ^* , each identically equal to ABC, to gether with a central square

Hence

sq on hypotenuse c=4rt $\angle^d \triangle^s$ + the central square = $4 \frac{1}{3}ab + (a-b)^2$ = $2ab + a^2 - 2ab + b^2$ = $a^2 + b^2$

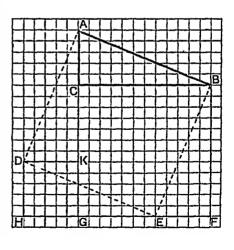


II Herc ABC is the given rt angled Δ, and the figs CF, HK are the sqq on CB, CA placed side by side

FE is made equal to DH or CA, and the two sqq CF, HK are cut along the lines BE, ED

Then it will be found that the \triangle DHE may be placed so as to fill up the space ACB, and the \triangle BFE may be made to fill the space AKD

Hence the two sqq CF, HK may be fitted together so as to form the single fig ABED, which will be found to be a perfect square, namely the square on the hypotenuse AB



FXERCISES

(Numerical and Graphical)

- 1. Draw a triangle ABC, right angled at C, having given
 - (1) a=3 em, b=4 em.
 - (11) $\alpha = 2.5 \text{ cm}$, b = 6.0 cm,
 - (m) a=1.2'', b=3.5''

In each case calculate the length of the hypotenuse c, and verify your result by incasurement

- 2. Draw a triangle ABC, right angled at C having given
 - (1) c=3 1, a=3 0, [See Problem 10]
 - (ii) c=5 3 cm, b=4 5 cm

In each case calculate the remaining side, and verify your result by measurement.

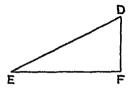
(The following examples are to be so'red by calculation, but in each case a plan should be drawn on some suitable wale, and the calculated result verified by measurement)

- 3 A ladder whose foot is 9 feet from the front of a house reaches to a window sill 40 feet above the ground. What is the length of the ladder?
- 4 A ship sails 33 miles due South, and then 56 miles due West How far is it then from its starting point
- 5 Two ships are observed from a signal station to bear respectively N E 60 km. distant, and N W 11 km distant. How far are they apart $^{\circ}$
- 6 A ladder 65 feet long reaches to a point in the face of a house 63 feet above the ground How far is the foot from the house?
- 7 B is due I at of A, but at an unknown distance C is due South of B, and distant 55 metres AC is known to be 73 metres. Find AB
- S A man travels 27 miles due South then 21 miles due West, finally 20 miles due North How far is he from his starting point?
- 9 From A go West 25 metres, then North 60 metres, then East 80 metres, finally South 12 metres How far are you then from A?
- 10 A ladder 50 feet long is placed so as to reach a window 48 feet high, and on turning the ladder over to the other side of the street, it reaches a point 14 feet high. Find the breadth of the street

THEOREM 30 [Euclid I 48]

If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle





Let ABC be a triangle in which the sq on AB = the sum of the sqq on BC, CA.

It is required to prove that ACB is a right angle

Make EF equal to BC

Draw FD perpr to EF, and make FD equal to CA.

Join ED

Proof

Because EF = BC, the sq on EF = the sq on BC And because FD = CA,

the sq on FD = the sq on CA

Hence the sum of the sqq on EF, FD = the sum of the sqq on BC, CA

But since EFD is a rt 4

And, by hypothesis, the sq on BC, CA = the sq on AB
the sq on DE = the sq on AB

DE = AB

the $\angle ACB = the \angle DFE$

Then in the A' ACB, DFE,

because $\begin{cases} AC = DF, \\ CB = FE, \\ and AB = DE, \end{cases}$

Theor 7

But, by construction, DFE is a right angle the \angle ACB is a right angle

QED

EVERCISES ON THFOREMS 29, 30

(Theoretical)

- 1 Shew that the square on the diagonal of a given square is double of the given square
- 2 In the $\triangle ABC$, AD is drawn perpendientar to the base BC If the side c is greater than b,

show that
$$c^2 - b^2 = BD^2 - DC^2$$

3 If from any point O within a triangle ABC, perpendiculars OX, OY, OZ are drawn to BC, CA, AB respectively—shew that

$$AZ^{2} + BX^{2} + CY^{2} = AY^{2} + CX^{2} + BZ^{2}$$

- 4 ABC is a triangle right nigled at A and the sides AB, AC are intersected by a straight line PQ, and BQ, PC are joined Prove that BQ²+PC²=BC²+PQ²
- 5 In a right-angled triangle four times the sum of the squares on the medians drawn from the acute angles is equal to five times the square on the hypotenuse
 - 6 Describe a square equal to the sum of two given squares
- 7 Describe a square equal to the difference between two given squares.
- 6 Divide a straight line into two parts so that the square on one part may be twice the square on the other
- 9 Divide a strught line into two parts such that the sum of their squares shall be equal to a given square

(Numerical and Graphical)

10 Determine which of the following triangles are right-angled

(i) a=14 cm, b=48 cm, c=50 em,

(ii) a=40 cm, b=10 cm, c=41 cm

(iii) a=20 cm, b=90 cm, c=101 cm

11 ABC is an isosceles triangle right angled at C, deduce from Theorem 29 that

AB=2AC

Illustrate this result graphically by drawing both diagonals of the square on AB, and one diagonal of the square on AC

If AC=BC=2", find AB to the nearest hundredth of an inch, and verify your calculation by actual construction and measurement

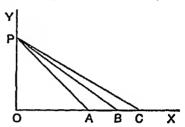
12. Draw a square on a diagonal of 6 cm. Calculate, and also measure, the length of a side. Find the area

PROBLEM 16

To draw squares whose areas shall be respectively twice, three-times, four-times, , that of a given square.

Hence find graphically approximate values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$,

Take OX, OY at right angles to one another, and from them mark off OA, OP, each one unit of length Join PA



Then
$$PA^2 = OP^2 + OA^2 = 1 + 1 = 2$$

$$PA = \sqrt{2}$$

From OX mark off OB equal to PA, and join PB, then PB² = $OP^2 + OB^2 = 1 + 2 = 3$

$$PB = \sqrt{3}$$

From OX mark off OC equal to PB, and join PC, then $PC^2 = OP^2 + OC^2 = 1 + 3 = 4$ $PC = \sqrt{4}$

The lengths of PA, PB, PC may now be found by measurement, and by continuing the process we may find $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$,

EXERCISES ON THEOREMS 29, 30 (Continued)

13 Prove the following formula

Diagonal of square= $*ide \times \sqrt{2}$

Hence find to the nearest centimetre the diagonal of a square on a side of 50 metres

Draw a plan (scale 1 cm to 10 metres) and obtain the result as nearly as you can by measurement

14 ABC is an equiliteral triangle of which each side =2m units, and the perpendicular from any vertex to the opposite side =p

Prove that $p=m\sqrt{3}$

Test this result graphically, when each side=8 cm.

15 If in a triangle $a=m^2-n^2$, b=2mn, $c=m^2+n^2$, prove algebraic ally that $c^2=a^2+b^2$

Hence by giving various numerical values to m and n, find sets of numbers representing the sides of right-angled triangles

- 16 In a triangle ABC, AD is drawn perpendicular to BC Let p denote the length of AD
 - (1) If a=25 cm, p=12 cm, BD=9 em, find b and c.
 - (n) If b=41'', c=50'', BD=30'', find p and a

And prove that $\sqrt{b-p^2} + \sqrt{c^2-p^2} = a$

17 In the triangle ABC, AD is drawn perpendicular to BC Prove that

$$c^2 - BD^2 = b^2 - CD^2$$

If a=51 em, b=20 cm, c=37 cm, find BD

Thence find p, the length of AD, and the area of the triangle ABC

- 18 Find by the method of the last example the areas of the triangles whose sides are as follows
- (1) $\alpha = 17$, b = 10, c = 9 (1) $\alpha = 25$ ft, b = 17 ft, c = 12 ft (11) $\alpha = 41$ cm, b = 28 cm, c = 15 cm (12) a = 40 yd, b = 37 yd, c = 13 yd.
- 19 A straight rod PQ slides between two straight rulers OX, OY placed at right angles to one another. In one position of the rod OP=56 cm, and OQ=33 cm. If in another position OP=40 cm, find OQ graphically, and test the accuracy of your drawing by calculation.
- 20 ABC is a triangle right-angled at C, and p is the length of the perpendicular from C on AB By expressing the area of the triangle in two ways, shew that

$$pc=ab$$

Hence deduce

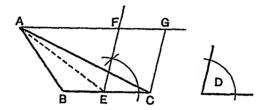
$$\frac{1}{1r} = \frac{1}{a} + \frac{1}{b}$$

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PROBLEMS ON AREAS

J PROBLEM 17

To describe a parallelogram equal to a given triangle, and having one of its angles equal to a given angle



Let ABC be the given triangle, and D the given angle

It is required to describe a parallelogram equal to ABC, and
having one of its angles equal to D

Construction.

Bisect BC at E

At E in CE, make the LCEF equal to D, through A draw AFG par' to BC, and through C draw CG par' to EF Then FECG is the required parm

Proof

Join AE

Now the \triangle ABE, AEC are on equal bases BE, EC, and of the same altitude,

the \triangle ABE = the \triangle AEC the \triangle ABC is double of the \triangle AEC

But FECG is a par™ by construction, and it is double of the △ AEC, being on the same base EC, and between the same par™ EC and AG

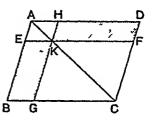
the par FECG = the \triangle ABC , and one of its angles, namely CEF, = the given \angle D

EXERCISES

(Graphical)

- 1 Draw a square on a side of 5 cm, and make a parallelogram of equal area on the same base, and having an augle of 45°
- Find (1) by calculation, (11) by measurement the length of an oblique side of the parallelogram.
- 2 Draw any parallelogram ABCD in which $AB=2\frac{1}{2}$ " and AD=2", and on the base AB draw a rhombus of equal area

DEFINITION In a parallelogram ABCD, if through any point K in the diagonal AC parallels EF, HG are drawn to the sides, then the figures EH, GF are called parallelograms about AC, and the figures EG, HF are said to be their complements



3 In the diagram of the preceding definition show by Theorem 21 that the complements EG, HF are equal in area

Hence, given a parallelogram EG, and a straight line HK, deduce a construction for driwing on HK as one side a parallelogram equal and equiangular to the parallelogram EG

4 Construct a rectangle equal in area to a given rectangle CDEF, and having one side equal to a given line AB.

If AB=6 cm, CD=8 cm, CF=3 cm, find by measurement the remaining side of the constructed rectangle

5 Given a parallelogram ABCD, in which AB=24'', AD=18'', and the $\angle A=55^\circ$ Construct au equiangular parallelogram of equal area, the greater side measuring 27'' Measure the shorter side

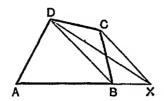
Repeat the process giving to A any other value, and compare your results What conclusion do you draw?

6 Draw a rectangle ou a side of 5 cm equal in area to au equilateral triaugle on a side of 6 cm

Measure the remaining side of the rectangle, and calculate its approximate area.

PROBLEM 18

To draw a triangle equal in area to a given quadrilateral.



Let ABCD be the given quadrilateral

It is required to describe a triangle equal to ABCD in area

Construction. Join DB
Through C draw CX part to DB, meeting AB produced in X.
Join DX

Then DAX is the required triangle

Proof Now the \triangle XDB, CDB are on the same base DB and between the same parts DB, CX,

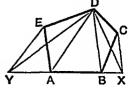
the $\triangle XDB =$ the $\triangle CDB$ in area

To each of these equals add the \triangle ADB, then the \triangle DAX = the fig ABCD

COROLLARY In the same way it is always possible to draw a reetilineal figure equal to a given reetilineal figure, and having fewer sides by one than the given figure, and thus step by step, any reetilineal figure may be reduced to a triangle of equal area.

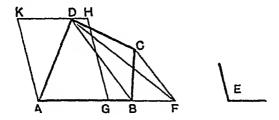
For example, in the adjoining diagram the five-sided fig EDCBA is equal in area to the four-sided fig EDXA

The fig EDXA may now be reduced to an equal \triangle DXY





To draw a parallelogram equal in area to a given rectilineal figure, and huving an angle equal to a given angle



Let ABCD be the given reetil fig, and E the given angle

It is required to draw a par equal to ABCD and having an angle equal to E

Construction Join DB
Through C draw CF par to DB, and meeting AB produced in F

Join DF

Then the △DAF = the fig ABCD Prob 18

Draw the par AGHK equal to the △ADF, and having the ∠KAG equal to the ∠E Prob 17

Then the par^m $KG = the \triangle ADF$ = the fig ABCD, and it has the $\angle KAG$ equal to the $\angle E$

Note If the given rectilineal figure has more than four sides, it must first be reduced, step by step, until it is replaced by an equivalent triangle.

EXERCISES

(Reduction of a Rectilineal Figure to an equivalent Triangle)

Draw a quadrilateral ABCD from the following data AB=BC=55 cm, CD=DA=45 cm, the $\triangle A=75^{\circ}$

Reduce the quadrilateral to a triangle of equal area. Measure the base and altitude of the triangle, and hence calculate the approximate area of the given figure.

2 Draw a quadrilateral ABCD having given

AB=28", BC=3.2", CD=33", DA=36", and the diagonal BD=30" Construct an equivalent triangle, and hence find the approximate area of the quadrilateral

V3 On a base AB, 4 cm in length, describe an equilateral pentagon (5 sides), having each of the angles at A and B 108°

Reduce the figure to a triangle of equal area, and by measuring its base and altitude, calculate the approximate area of the pentagon.

A quadrilateral field ABCD has the following measurements AB=450 metres, BC=380 m, CD=330 m, AD=390 m., and the diagonal AC=660 m

Draw a plan (scale 1 cm to 50 metres) Reduce your plan to an equivalent triangle, and measure its base and altitude Hence estimate the area of the field

(Problems State your construction, and give a theoretical proof)

- 5 Reduce a triangle ABC to a triangle of equal area having its base BD of given length (D lies in BC, or BC produced.)
- 6 Construct a triangle equal in area to a given triangle, and having a given altitude
- 7 ABC is a given triangle, and X a given point. Draw a triangle equal in area to ABC, having its vertex at X, and its base in the same straight line as BC.
- 8 Construct a triangle equal in area to the quadrilateral ABCD, having its vertex at a given point X in DC, and its base in the same straight line as AB
- 9 Shew how a triangle may be divided into n equal parts by straight lines drawn through one of its angular points

10 Bisect a triangle by a straight line drawn through a given point in one of its sides

[Let ABC be the given \triangle , and P the given point in the side AB

Bisect AB at Z, and join CZ, CP Through Z draw ZQ parallel to CP Join PQ. Then PQ bisects the Δ ?

CP Z P Q

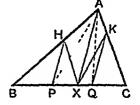
Il Trisect a triangle by straight lines drawn from a given point in one of its sides

[Let ABC be the given △, and X the given point in the side BC

Trisect BC at the points P, Q. Prob 7
Join AX, and through P and Q draw PH and
QK parallel to AX

Join XH, XK

These straight lines trisect the \triangle , as may be shewn by joining AP, AQ]



- 12 Cut off from a given triangle a fourth, fifth, sixth, or any part required by a straight line drawn from a given point in one of its sides
- 13 Bisect a quadrilateral by a straight line drawn through an angular point

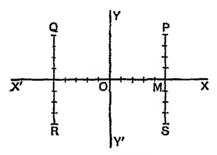
[Reduce the quadrilateral to a triangle of equal area, and join the vertex to the middle point of the base]

14 Cut off from a given quadrilateral a third, a fourth a fifth, or any part required, by a straight line drawn through a given angular point

AXES OF REFERENCE COORDINATES

EXERCISES FOR SQUARED PAPER

If we take two fixed straight lines XOX', YOY' cutting one another at right angles at O, the position of any point P with reference to these lines is known when we know its distances from each of them. Such lines are called axes of reference, XOX' being known as the axis of x, and YOY' as the axis of y. Their point of intersection O is called the origin.



The lines XOX', YOY' are usually drawn horizontally and vertically

In practice the distances of P from the axes are estimated

thus

From P, PM is drawn perpendicular to X'X, and OM and PM are measured

OM is called the abscissa of the point P, and is denoted by x PM ... ordinate ... y

The abscissa and ordinate taken together are called the coordinates of the point P, and are denoted by (x, y)

We may thus find the position of a point if its coordinates are known

EXAMPLE Plot the point whose coordinates are (5, 4)
Along OX mark off OM, 5 units in length

At M draw MP perp to OX, making MP 4 units in length. Then P is the point whose coordinates are (5, 4)

The axes of reference divide the plane into four regions XOY, YOX', X'OY', Y'OX, known respectively as the first, second, unird, and fourth quadrants

It is clear that in each quadrant there is a point whose distances from the axes are equal to those of P in the above diagram, namely, 5 units and 4 units

The coordinates of these points are distinguished by the use of the positive and negative signs, according to the following

system

Abscisse measured along the x-axis to the right of the origin are positive, those measured to the left of the origin are negative. Ordinates which he above the x-axis (that is, in the first and second quadrants) are positive, those which he below the x-axis (that is, in the third and fourth quadrants) are negative.

Thus the coordinates of the points Q, R, S are (-5, 4), (-5, -4), and (5, -4) respectively

Note. The coordinates of the origin are (0, 0)

In practice it is convenient to use squared paper. Two intersecting lines should be chosen as axes, and slightly thickened to aid the eye, then one or more of the length divisions may be taken as the linear unit. The paper used in the following examples is ruled to tenths of an inch

EXAMPLE 1 The coordinates of the points A and B are (7, 8) and (-5, 3) plot the points and find the distance between them.

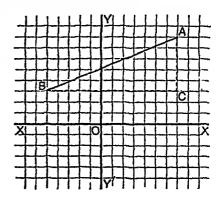
After plotting the points as in the diagram, we may find AB approximately by direct measurement.

Or we may proceed thus

Draw through B a line part to XX' to meet the ordinate of A at C. Then ACB is a rt angled \triangle in which BC=12, and AC=5

Now
$$AB^2 = BC^2 + AC^2$$

= $12^2 + 5^2$
= $144 + 25$
= 169
• $AB = 13$

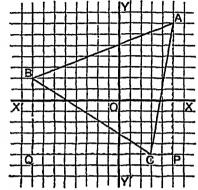


XX'

Example 2 The coordinates of A, B, and C are (5, 7), (-8, 2), and (3, -5), plot these points and find the area of the triangle of which these are the vertices

Having plotted the points as in the diagram, we may measure AB, and draw and measure the perp from C on AB Hence the approximate area may be calculated.

Or we may proceed thus
Through A and B draw AP, BQ
par' to YY'
Through C draw PQ par' to



Then the \triangle ABC = the trap^m APQB - the two rt angled \triangle APC, BQC = $\frac{1}{2}$ PQ(AP+BQ) - $\frac{1}{2}$ AP PC - $\frac{1}{3}$ BQ QC = $\frac{1}{2} \times 13 \times 19$ - $\frac{1}{2} \times 12 \times 2$ - $\frac{1}{3} \times 7 \times 11$ = 73 units of area

EXERCISES

1. Plot the following sets of points

(1)
$$(6, 4)$$
, $(-6, 4)$, $(-6, -4)$, $(6, -4)$,

(u) (8, 0), (0, 8), (-8, 0), (0, -8),

(m) (12, 5), (5, 12), (-12, 5), (-5, 12)

2 Plot the following points, and show experimentally that each set he in one straight line

(1)
$$(9, 7), (0, 0), (-9, -7),$$
 (11) $(-9, 7), (0, 0), (9, -7)$

Explain these results theoretically

3 Plot the following pairs of points, join the points in each case, and measure the coordinates of the mid-point of the joining line.

Shew why in each case the coordinates of the mid point are respectively half the sum of the abscissæ and half the sum of the ordinates of the given points

4 Plot the following pairs of points, and find the coordinates of the mid point of their joining lines

(1) (0, 0), (8, 10),

(11) (8, 0), (0, 10),

(iii) (0, 0), (-8, -10),

(iv) (-8, 0), (0, -10)

- 5 Find the coordinates of the points of triscetion of the line joining (0, 0) to (18, 15)
 - 6 Plot the two following series of points

(1)
$$(5, 0)$$
, $(5, 2)$, $(5, 5)$, $(5, -1)$, $(5, -4)$,

(11)
$$(-4, 8)$$
, $(-1, 8)$, $(0, 8)$, $(3, 8)$, $(6, 8)$

Show that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the coordinates of the point in which they intersect

- 7 Plot the following points, and calculate their distances from the origin
 - (1) (15, 8), (1) (-15, -8), (11) (24'', 7''), (12) (-7'', 24'')Check your results by measurement
- 8 Plot the following pairs of points, and in each case calculate the distance between them
 - (1) (4, 0), (0, 3), (11) (9, 8), (5, 5),
 - (m) (15, 0), (0, 8), (1v) (10, 4), (-5, 12),
 - (1) (20, 12), (-15, 0), (11), (20, 9), (-15, -3)

Verify your calculation by measurement

- 9 Shew that the points (-3, 2), (3, 10), (7, 2) are the angular points of an isosecles triangle. Calculate and measure the lengths of the equal sides
- 10 Plot the eight points (0, 5), (3, 4), (5, 0), (4, -3), (-5, 0), (0, -5), (-4, 3), (-4, -3), and show that they all he on a circle whose centre is the origin
- 11 Explain by a diagram why the distances between the following pairs of points are all equal
 - (1) (a, 0), (0, b), (11) (b, 0), (0, a), (111) (0, 0), (a, b)
 - 12. Draw the straight lines joining
- (1) $(\alpha, 0)$ and $(0, \alpha)$, (11) (0, 0) and (α, α) , and prove that these lines bisect each other at right angles
- 13 Shew that (0, 4), (12, 9), (12, -4) are the vertices of an isosceles triangle whose base is bisected by the axis of x
- 14 Three vertices of a rectangle are (14, 0), (14, 10), and (0, 10) find the coordinates of the fourth vertex, and of the intersection of the diagonals
- 15 Prove that the four points (0, 0), (13, 0) (18, 12), (5, 12) are the angular points of a rhombus Find the length of each side, and the coordinates of the intersection of the diagonals
- 16 Plot the locus of a point which moves so that its distances from the points (0, 0) and (4, -4) are always equal to one another Where does the locus cut the axes.

- Shew that the following groups of points are the vertices of gles. Draw the figures, and calculate their areas rectangles
 - (1) (4, 3), (17, 3), (17, 12), (4, 12),
 - (11) (3, 2), (3, 15), (-6, 15), (-6, 2),
 - (iii) (5, 1), (-8, 1), (-8, -8), (5, -8).
- 18 Join in order the points (1'', 0), (0 1''), (-1'', 0), (0, -1'') what kind is the quadrilateral so formed 9 Find its area

If a second figure is formed by joining the middle points of the first, find its area

- Plot the triangles given by the following sets of points, and find their areas.
 - (1) (10, 10), (4, 0), (18, 0), (n) (10, -10), (4, 0), (18, 0).
- (111) (-10, 10), (-4, 0), (-18, 0), (1v) (-10, -10), (-4, 0), (-18, 0)
 - Draw the triangles given by the points 20
- (1) (0, 0), (5, 3), (6, 0). (II) (0, 0), (3, 0), (0, 6) Find their areas, and measure the angles of the first triangle
- Plot the triangles given by the following sets of points that in each case one side is parallel to one of the axes Hence find the
- (1) (0, 0), (12, 10), (12, -6), (11) (0, 0), (5, 8), (-15, 8), (11) (0, 0), (-12, 12), (-12, -8), (12) (0, 0), (-6, -8), (-13, 8), (13)
- 22 In the following triangles shew that two sides of each are parallel to the axes Find their areas
- (11) (8, 3), (8, 18), (0, 18). (1) (5, 5), (15, 5), (15, 15)(iii) (4, 8), (-16, -4), (4, -4), (iv) (1, 15), (-11, 15), (1, -7)
- Shew that (-5, 5), (7, 10), (10, 6), (-2, 1) are the angular points of a parallelogram. Find its sides and area
- Shew that each of the following sets of points gives a trapezium Find the area of each
 - (1) (3,0), (3,3), (9,0), (9,6), (11) (9,3), (-5,3), (-2,-3), (0,-3),
- (m) (8,4), (4,4), (11,-1), (3,-1), (17) (0,0), (-1,5), (-4,5), (-8,0)
 - Find the area of the triangles given by the following points
- (1) (5, 5), (20, 10), (12, 14), (11) (7, 6), (-10, 4), (-4, -3); (iii) (0, -6), (0, -3), (14, 5), (iv) (6, 4), (-7, -6), (-2, -15)
- Shew that (-5, 0), (7, 5), (19, 0), (7, -5) are the angular points of a rhombus Find its sides and its area

- Join the points (0, -5), (12, 0), (4, 6), (-8, -3), in the order given Calculate the lengths of the first three sides and measure the fourth Find the areas of the portions of the figure lying in the first and fourth quadrants.
 - 28 The coordinates of four points A, B, C, D are respectively

$$(-4, -4), (-10, 4), (-10, 13) (5, 5)$$

Calculate the lengths of AB, BC, CD, and measure AD Also calculate the area of ABCD by considering it as the difference of two triangles

29 Draw the figure whose angular points are given by (0, -3), (8, 3), (-4, 8), (-4, 3), (0, 0)

Find the lengths of its sides, taking the points in the above order Also divide it into three right-ingled triangles, and hence find its area

- 30 A plan of a triangular field ABC is drawn on squared paper (scale $1^n=100$ yds) On the plan the coordinates of A, B, C are $(-1^n, -3^n)$, $(3^n, 4^n)$, $(-5^n, -2^n)$ respectively. Find the area of the field, the length of the side represented by BC, and the distance from this side of the opposite corner of the field
- 31 Shew that the points (6, 0), (20, 6), (14, 20), (0, 14) are the vertices of a square. Measure a side and hence find the approximate area. Calculate the area exactly (1) by drawing a circumscribing square through its vertices, (11) by subdividing the given square as in the first figure on page 120

MISCELLANEOUS EXERCISES

- 1 AB and AC are unequal sides of a triangle ABC, AX is the median through A, AP biscots the angle BAC, and AD is the perpendicular from A to BC $\,$ Prove that AP is intermediate in position and magnitude to AX and AD
- 2 In a triangle if a perpendicular is drawn from one extremity of the base to the bisector of the vertical angle, (1) it will make with either of the sides containing the vertical angle an angle equal to half the sum of the angles at the base, (11) it will make with the base an augle equal to half the difference of the angles at the base
- 3 In any triangle the angle contained by the bisector of the vertical angle and the perpendicular from the vertex to the base is equal to half the difference of the angles at the base.
- 4 Construct a right-angled triangle having given the hypotenuse and the difference of the other sides
- 5 Construct a triangle, having given the base, the difference of the angles at the base, and (i) the difference, (ii) the sum of the remaining sides
- 6 Construct an isosceles triangle, having given the base and the sum of one of the equal sides and the perpendicular from the vertex to the base
- 7 Shew how to divide a given straight line so that the square on one part may be double of the square on the other
- 8 ABCD is a parallelogram, and O is any point without the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the sum of the triangles OAD, OAB
- If O is within the angle BAD or its opposite vertical angle, shew that the triangle OAC is equal to the difference of the triangles OAD, OAB
- 9 The area of a quadrilateral is equal to the area of a triangle having two of its sides equal to the diagonals of the given figure, and the included angle equal to either of the angles between the diagonals.
- 10 Find the locus of the intersection of the medians of triangles described on a given base and of given area.
- 11 On the base of a given triangle construct a second triangle equal in area to the first, and having its vertex in a given straight line
- 12 ABCD is a parallelogram made of rods connected by hinges I AB is fixed, find the locus of the middle point of CD

ANSWERS TO NUMERICAL EXERCISES

Since the ilmost care cannot ensure a solute accuracy in graylucal work, results to educate are likely to be only approximate. The annurs here given are those found by calculation, and being true to far as they go, furnish a standard by which the student may ter the correctness of his drawing and measurement. Results within one per cent of those given in the Ansiers was usually be considered satisfactors.

Exercises Page 15

- 1 30°, 126°, 261°, 85° 11 mm., 37 mm.
- 2 1122°, 155°, 5 hrs 45 min 3 50°, 8 hrs 40 min.
- 4. (1) 145°, 35°, 145° (11) 55°, 55° 86°, 94°

Exercises Page 27

- 1 68°, 37°, 75° v nearly 2 60 cm 4 2.2°, 50°, 73° nearly
- 5 37 ft 6 101 metres 7 27 ft 8 424 yds , nearly , N W.
- 9 281 yds, 155 yds, 153 yds 10 214 yds

Exercises Page 41

1 125°, 55°, 125° 12 15 secs , 30 secs

Exercises Page 43

3 21° 4. 27° 5 92°, 46° 6 67°, 62°

Exercises Page 45

- 1 30°, 60°, 90° 2 (1) 36°, 72°, 72°, (11) 20°, 80°, 80°
- 3 40° 4 51°, 111°, 15 5 (1) 34°, (11) 107° 6 65 7 120° 8 36°, 72°, 108°, 144°
- 6 68 7 120° 9 165° 11 5. 15

Exercises Page 47

2 (1) 45°, (11) 36° 3 (1) 12, (11) 15

Exercises Page 54.

4. (1) 81°, c. (11) 55°

10	Degrees	15	30	45°	Cũ.	75°
	Cm.	4 1	46	57	8-0	15 6

11	Degrees	0,	30°	60,	3 0,,	120	150°	180°
	Cm	1.0	20	36	50	61	68	70

12 37 ft. 13 112 ft. 14 346 yds 693 yds.

Exercises Page 61

54°, 72°, 54° 15 380 16 4 14.

(1) 16, (11) 45°, (111) 114° per sec 18

Exercises Page 68

2-24" 4 0 39 5 2 54 8 10 6 cm. 2 6 80 cm 3

9 20 milcs. 12 6 km 3 35" 10

147 miles, 235 km 1 cm represents 22 km 11

1" represents 15 mi , 1" represents 20 mi 12

Page 79 Exercises

5 24" 3 0 53 m 4. 1 3 cm.

Exercises Page 84.

43 cm, 52 cm, 61 cm 2 1 10 200 yards 1 3

4 65°, 77 m, 61 m, 56 m 5 6 04 knots S, 15° E, nearly

7 43 cm, 98 cm, 60°, 120° 6 Results equal. 9 cm 8

(1) One solution, (11) two, (111) one, right-angled, (1v) impossible

9 380 yds 10 65 cm 11 69 cm

Two solutions, 104 cm or 45 cm $28\,\mathrm{cm}$, $45\,\mathrm{cm}$, $53\,\mathrm{cm}$ 12 16

58 cm . 42 cm. 18 19 7 cm . 8 cm

Page 89 Exercises

60°, 120° 2 3 54" 1 3 2 12" 4 44 cm (1) 4.25", (11) B=D=90° 5 16 4 cm , 3 4% 90° 6 7

Exercises Page 102

3 1 6 sq m 2 6 sq m 280 sq in 4. 350 sq in 3 30 sq in 3 36 sq in 5 ថ

198 sq m 8 42 sq ft 7 9 10,000 sq m 10 110 sq ft 11 5 cm. 12 26 m

14. 900 sq yds , 48 yds , 48" 15 11700 sq m

16 1 cm = 10 vds17 16" 18 600 sq ft 19 1152 sq ft.

100 sq ft 20 21 156 sq ft 110 sq ft 22

28 288 sq ft 24. 72 sq ft 25 75 sq ft

Page 105 Exercises

34 sq 1n (1) 22 cm , (11) 3 6" 2 8 574 5 sq in 4.

1.93", 75° 1 5" Б

Exercises Page 107

- (1) 180 sq ft , (11) 8 4 sq in , 1 hectare 1
- 2 (1) 13 44 sq cm , (11) 15 40 sq cm , (111) 20 50 sq cm
- 4. 63 sq in 3 15 sq cm 5
 - (1) 8", (11) 13 cm 3 36 sq in 6

Exercises Page 110

- 11400 sq yds 2 6312 sq m 1
- 2 04", 2 20" 24 cm , 51 cm 4 3

5	Angle	0°	30°	60°	90°	120°	150°	180°
	Area in sq cm	0	75	13 0	15 0	13 0	75	0

Exercises Page 111

- 84 sq yds 1 66 sq ft 2 3 126 sq m 306 sq m 6
- 5 180 sq ft 4. 132 sq cm

Exercises Page 113

- 170 sq ft 3 615 sq m 6 sq in 2 4 84 sq m 1 7 24 sq cm
- 5 31-2 sq cm 6 5.20 sq in

Exercises Page 115

- 1 (1) 25 5 sq cm, (11) 15 6 sq cm
- 2 (1) 8.95 sq in , (11) 9.5 sq in 3 12500 sq m

Exercises Page 116

4. 33 sq m 5 75 cm 6 36 sa m

Exercises Page 121

- (1) 5 cm , (11) 6 5 cm , (111) 3 7" 2 (1) 1 6", (11) 2 8 cm 1
- 4. 65 miles 5 6 l km 41 ft 6 16 ft 3
- 25 miles 9 73 m 8 10 62 ft 7 48 m

Exercises Page 123

- (1) and (111) 11 2 83" 70 71 sq m 14 p=6 93 cm 12 4-24 cm, 18 sq cm 10
- 13
- (1) 20 cm , 15 cm (11) 40 cm , 39 cm 16
- 17 35 cm , 12 cm , 306 sq cm
- (1) 36 sq in , (11) 90 sq ft , (111) 126 sq cm , (1v) 240 sq yds 18
- 51 cm nearly 19

31

```
Page 127
                      Exercises
                                  16" 6 31 cm . 156 sq cm
1
    71 cm 4 40 cm
                              5
                     Exercises Page 130
                                    8 40 sq in
    23.90 sq cm
1
                                 2
3
   27 52 sq cm
                                     129800 sq m
                     Exercises Page 134.
    (1) (8, 5), (11) (10, 10)
3
    (1) (4, 5), (11) (4, 5), (111) (-4, -5), (117) (-4, -5)
4
                              6 (5, 8)
5
    (6, 5), (12, 10)
    (1) 17, (11) 17, (111) 25°, 25"
7
    (1) and (11) 5, (111) and (1v) 17, (1) and (v1) 37
                                                         10
8
                                                      9
                              15
14.
                                  13. (9, 6)
    (0, 0) (7, 5)
    A straight line passing through the points (4, 0), (0, -4)
16
                                  18 Asquare 2 sq in 1 sq in
17
    117 units of area in each case
                                      9 units of area. 31°, 71°, 78°
19
    Each = 70 units of area
                                  20
21
    (1) 96, (11) 80, (111) 120, (1v) 104
22
    (1) 50, (11) 60, (111) 120, (1v) 132
    Sides 5, 13; area 63 24. (1) 27, (11) 21, (111) 30, (11) 27 5
23
25
    (1) 50, (11) 65 5, (111) 21, (11) 83 5
26
    Each side 13, area 120 27 13, 10, 15, 8 24, 42, 30
28
    AB=10, BC=9, CD=17, DA=127 Area=130 5
29
    10, 13, 5, 5, 3 Area = 60 30 160,000 sq yds 1000 yds 320 yds
```

¥Raj Bahadui MATHUR.≫

Side=15 23, area=232 units of area

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